Studies on Application of a Markov Approximation Method to Structural Reliability Analyses

Takeyuki Tanaka

December, 1994

Abstract

Some topics in the structural reliability analysis are investigated from the stand point of the random fatigue propagation with use of a Markov approximation method.

Firstly, the structural reliability assurance by in-service inspection is argued on two different policies, one is a repeated inspection with fixed intervals, and the other is a history-dependent inspection the intervals of which depend on the results of past inspections. The crack growth model is expressed as a stochastic differential equation which is randomized by the process of the external loading and the crack propagation resistance. In order to investigate the effect of the inspections, a method to evaluate the failure probability of the structural component is presented on the assumption that the component is exchanged when cracks are detected. One of the results shows an optimal inspection interval to minimize the failure probability in the service period. Under the same assumption, the efficiency of history-dependent inspection is argued. The failure probability and the mean inspection times are calculated by the idea from a renewal procedure. The results show that the inspection intervals has a remarkable influence on the structural reliability of components as well as the randomness in the growth process has.

Secondary, the random propagation of two collinear cracks is investigated in the context of multiple site damage. They are expressed by the two dimensional stochastic process which mutually accelerate the growth rate of each crack. A method for evaluating a joint probability density function is proposed with the aid of both the analytic approximation and the numerical analysis. The results are applied to the calculation of the probability that the two cracks meet within a certain time interval.

Thirdly, the random propagation of semi-elliptical surface crack is investigated as a bivariate stochastic process. Under some assumptions, an approximation method is introduced to derive the joint probability density function for crack length with respect to the depth and to the surface direction. The result is applied to the Leak Before Break (LBB) assessment in the piping system, where the probability of LBB occurence and the hazard rate are calculated.

Finally, the random crack propagation including random overloads is argued. The retardation effect by overload is taken into account in the distribution function of the crack length. The method to evaluate the reliability and the hazard rate is proposed on the basis of the two-criteria approach. The fast Monte-Carlo simulation technique, known as an importance sampling method, is applied to the numerical calculation to evaluate the failure probability in the multi dimensional space.

Acknowledgments

The theme of this study arose from many discussions with Dr. Hiroaki Tanaka and the late professor Akira Tsurui. I would also like to thank professors Chiaki Ihara, Akira Ueda and Yasuo Matsushita for their continuous encouragement. Especially, I express my greatest gratitude to professor Toyonori Munakata for his helpful advice on my works.

Table of Contents

Chapter I

Introduction

In 1954, a serious aircraft accident happened. The aircraft G–ALYP, known as COMET–I type, of the BOAC suddenly broke up to pieces in the air and fell in the sea near Elbe Island. The same kind of the accidents on the same type aircraft happened three times, while the cause of the accidents was left unknown for a long time. After the long time investigation, it was clarified that the accidents are caused by the cracks which sited at the edge of the square window. This is the well–known accident caused by the structural failure due to material fatigue. The material fatigue often occurs not only in aircrafts but also ships, bridges, pressure vessels, piping, and so on. In Japan, the crash of the Boeing 747SR of JAL, the break of the piping system in the atomic generator plant at Mihama are still fresh in our memory. As is seen in them, the fatigue failure often becomes a cause of the fracture of the aged structure.

Fatigue failure is a phenomenon that the crack in the material grows under the repeated loads, at last, to cause the failure. It is a matter of course that the material having the large crack easily breaks down by much weaker tension than the static strength of the material. Generally, the tough material, such as metal or ceramics, is weak against the fatigue crack because the intensity of stress concentrates at the edge of the crack by external tension. According to fracture mechanics, the stress intensity σ is estimated as

$$\sigma = \frac{K}{\sqrt{r}} g(\theta) \tag{1.1}$$

near the edge of the crack, where r is a distance from the edge, $g(\theta)$ is a geometry factor of the crack against the direction of the external tension, and K, so called the stress intensity factor, is a function of the crack length and the intensity of the external tension. In the most familiar example of center penetrated crack in the infinity body, K is given as

$$K = S\sqrt{\pi a}\,, \tag{1.2}$$

where a and S are the half size of the crack length and a external tension stress, respectively. It is commonly assumed that the material fails if the magnitude of K exceeds a certain limit K_c, called fracture toughness.

In the fatigue failure, the crack grows under the much less level of K than K_c, so that the stress level should be kept in low level. Moreover, it is important to know the life of the material up to failure. Among several models of the crack propagation proposed by different authors, the following Paris-Erdogan's law is well-known :

$$\frac{da}{dn} = C(\Delta K)^m, \tag{1.3}$$

where n is the number of loading cycles, ΔK is a range of the varying stress intensity factor, and, C and m are material constant. Paris-Erdogan's law is an empirical expression, which is perceived to hold in a certain range of a. Therefore, if an initial crack length is known, the residual life is estimated from the integral of the above equation.

However, it is a well known fact that the actual data on strength of metallic materials is often scattered considerably. The variance of the strength is caused not only by the inhomogeneity in materials, such as a defect of the crystal grain, but also by the flaws squeezed through manufacturing processes of products. These imperfections are, in actual cases, quite difficult to be perfectly controlled. In addition, the external stress also exhibits randomness and uncertainty in both temporal and spatial change, which arises from natural phenomena such as winds, waves, temperature, etc., as well as the artificial irregularity of operating processes, for instance, a random vibration of internal pressure of boiler vessels.

In order to assure the safety of structures under these random circumstances, so-called safety factor, which is a ratio of the allowable load to the material strength, has been practically used in the structural design. The safety factor often assigned by experience, but we cannot always find its theoretical foundations. In 1947, Freudenthal[1,2] criticized the safety factor from a view point of the probability theory, and introduced a concept of reliability engineering into the safety design, which became a starting point of the structural reliability design in the present day. Treating load carrying capacity and load as random variables, he proposed the way of calculating the probability that the load exceeds the strength, that is, the probability of failure, and also proposed that the safety factor should be determined so as to make the failure probability smaller than a certain maximum allowable value.

Originally, reliability engineering was concerned with the quality warranty of the mass products such as electronic parts. Structural reliability design is nothing but an idea to evaluate the safety of the structure by the methodology of the reliability engineering. The same concept is used in the structural reliability, for instance, reliability function, hazard rate, life distribution, and so on. In comparison to the traditional reliability theory for mass products, the structural reliability analysis has some different characteristics as listed in the following items.

(1) The traditional reliability theory for mass products, the reliability function is often derived by a mathematical model such as Poisson distribution. In the case of the structural reliability problem, however, the presumption of the model is generally impossible, since large scale structures are not mass-produced, and their service environments are frequently influenced by uncertain natural phenomena.

(2) Generally, since the failure of structures is closely concerned with loss of human life, the reliability of each component is required to be extremely high compared with that of electronic system.

(3) In structures, we need to go into detailed contents of failure rate, and to control its value by similar reason as item (2).

Among them, a special attention should be paid to item (2) in the assessment of structural reliability. Because of this requirement, the reliability of each component must be evaluated with very high accuracy and confidence. Generally in computer simulation technique, it will be very hard to determine the detail of probability distribution curve in the high reliability region. Therefore, the analytical method without using computer simulation has a very important meaning especially in structural reliability engineering.

On the other hand, item (3) suggests that physical parameters should be reflected in a model for analytical research. There are, roughly speaking, two kinds of stochastic models to investigate random phenomena, one is the mathematical stochastic model and the other is the physical stochastic model. The mathematical model has been widely used in reliability assurance for electronic systems. In order to achieve the purpose above, however, the physical model has to be used rather than the mathematical one, since it would be difficult to introduce physical quantities naturally into the mathematical model.

As such, the structural reliability engineering always requires reasonable blending between the strict analysis and the compatibility with physical laws obtained empirically or theoretically. Therefore, we have to keep in mind such a requirement in the evaluation of the structural reliability.

Hereafter, we will make a brief survey of the studies on the random fatigue crack propagation. The random fatigue is a part of the major problem in structural reliability.

Although there can be many kinds of failure type, for example, sudden brittle fractured, wear-out failure, corrosion cracking, etc., the most part of accidents in real structures is shared by fatigue fracture. Hence, we are chiefly concerned with the safety assessment against the fatigue failure. In recent years, a damage tolerance principle is applied to the design of machines and structures. Under the damage tolerant design, it is a matter of cource that exact understanding of the properties of fatigue phenomena is one of the most important themes in the structural reliability engineering.

Since the fatigue failure is caused by the crack propagation under the repeated loads, quantitative analysis of fatigue crack growth processes has to be carried out to assess the safety against the fatigue failure. Because of many uncertainties such as material properties and loads under the variety of surroundings as mentioned above, the processes generally exhibit random characters. Therefore, the randomness has to be clarified in the structural reliability analysis.

In this respect, we must care about the fact that the fatigue is a degrading process, that is to say, the strength of structural component gradually decreases as time elapses. Hence, the time-dependent reliability analysis must be performed by comprehending fatigue phenomenon as a degrading process.

The first study that treats the fatigue crack propagation process as a stochastic process is a Markov chain model proposed by Bogdanoff in 1978. Its outline is described below [3-7].

In the stochastic model constructed by Bogdanoff, the state of damage d is assumed to be a discrete variable labeled by 1, 2, 3, ..., b, in which the final state b corresponds to replacement or failure. He considered that the damage caused by loading statistically accumulates, and that the state d proceeds to the final state b non-decreasingly. The transition is assumed to have a Markov property by disregarding the transitions during the so-called duty cycle. That is to say, an idea of imbedded Markov chain was applied to the modeling of the random crack propagation. According to these assumptions, setting up suitable values for transition probability, Bogdanoff pointed out that the probability distribution for the fatigue crack propagation life due to the material inhomogeneity can be, although qualitatively, reproduced.

Bogdanoff's theory is worthy to be notice in the point that Markov approximation concept is introduced into the investigation of the crack propagation processes for the first time. However, his theory contains the following weaknesses;

(1) Since both of state and time variables are discrete, the mathematical treatment becomes frequently complicated and it would be difficult to evaluate the probability distribution with high accuracy.

(2) Since Bogdanoff's model is purely mathematical, parameters appearing in his model such as transition probability are not clearly related to physical quantities whose values are clarified through experiments.

(3) When the stress amplitude is varied, parameters must be re-evaluate. Therefore, his model will not fit to an analysis under spectrum loading.

After the proposition of the Markov chain model by Bogdanoff, Lin and Yang reported a study on the statistical properties of fatigue crack propagation in 1983 [8]. In their study, they applied the theory of stochastic differential equation to the fatigue process. The survey of their study is as follows.

The crack propagation law can be generally expressed as follows;

$$\frac{da}{dn} = Cf(K, \Delta K, S, R, a), \tag{1.4}$$

where a represents a crack length after n-cycles of loading, K a stress intensity factor, ΔK its variation range, S a stress amplitude, and R a stress ratio, respectively. For instance, Paris-Erdogan's fatigue crack propagation law [9], which is most widely accepted today, is expressed as

$$\frac{da}{dn} = C(\Delta K)^m. \tag{1.5}$$

Lin and Yang randomized the above equation as

$$\frac{da}{dn} = \{C + Y(n)\} f(K, \Delta K, S, R, a), \tag{1.6}$$

where $Y(n)$ represents a stationary stochastic process with mean zero, according to the idea of Markov approximation proposed by Stratonovich. They considered that the growth process $a(n)$ can be regarded approximately as a Markov process by giving attention only to the variation in the time interval larger than the correlation time of $Y(n)$.

Further, under the assumption that Eq.(1.6) holds as a stochastic differential equation, they gave a method to evaluate statistical moments of the time when the crack grows to a certain critical crack length, what we call the life of the component.

Lin and Yang's method follows the Bogdanoff's one in the point that the crack propagation process is treated as a Markov process, while it differs from the

Bogdanoff's one in the point that the empirically obtained propagation law is randomized, that is a physical stochastic model is introduced. Although the relationship between physical quantities and the statistical properties of $Y(n)$ are not clearly indicated in their study, their method seems to go a step towards the practical requirement. However, it must be noticed that it includes the following serious controversial point.

From substitution of the simple expression of K in Eq. (1.2) into the Paris's law (1.3), we get a differential equation for the crack growth:

$$\frac{da}{dn} = C(\Delta S)^m a^{m/2} \tag{1.7}$$

One can see that the above equation is easily solved. It is obvious that it grows rapidly and diverges in a finite time region for n if m > 2. According to the results of experiments, this situation happens for most of metallic materials. This fact leads the several difficulty on the randomization of the equation. For example, the statistical moments of the crack length never exist in an ordinary sense. That is already pointed out by Bogdanoff [7]. He proved that, if we treat the crack length as a dependent variable then its statistical moments do not exist, while if we treat the number of cycles of loading as a dependent variable then its statistical moments exist. In the study of Lin and Yang, the difficulty above was not overcome, and their trial to calculate a probability distribution of crack length was not successful.

In 1984, Tsurui and Ishikawa proposed the solution to this difficulty by extending the Markov approximation to such an explosive process [11]. Giving attention to the fact that the crack propagation resistance C takes on a sufficiently small positive value, they investigated a vector process $X(t)$ that grows according to the following stochastic differential equation :

$$\frac{dX}{dt} = \varepsilon F(S(t), X), \tag{1.8}$$

where X, t and ε correspond to the crack length, the loading cycles and the crack growthe resistance, respectively. The function F is a known positived-valued deterministic function. This type of the stochastic differential equation was first studied by Stratonovich [10]. Generally, the solution process $X(t)$ does not always have Markov property, but can be treated as a Markov process approximately if we pay attention only to an increment process, the time intervals of which are sufficiently large to be compared with the correlation time of the external process $S(t)$. Under this assumption and the perturbation method with respect to the small parameter ε, the transition probability density function $w(x,t|x_0)$ such that

$$w(x,t|x_0)dx = \Pr[x \le X(t) < x + dx | X(0) = x_0] \tag{1.9}$$

is given as a solution of the following partial differential equation :

$$\frac{\partial w}{\partial t} = -\varepsilon \frac{\partial}{\partial x}\{M(x,t)w\} + \frac{\varepsilon}{2}\frac{\partial^2}{\partial x^2}\{D(x,t)w\}, \tag{1.10}$$

which is usually known as a Fokker-Planck equation, where,

$$\begin{aligned} M(x,t) = E^*[F(S(t),x)] + \varepsilon\int_{-\infty}^{0}\Big\{ & E^*\left[\frac{\partial F(S(t),x)}{\partial x}F(S(t+t',x)\right] \\ & - \frac{\partial E^*[F(S(t),x)]}{\partial x}E^*[F(S(t+t',x)]\Big\}dt', \end{aligned} \tag{1.11}$$

$$\begin{aligned} D(x,t) = 2\varepsilon\int_{-\infty}^{0}\{ & E^*[F(S(t),x)F(S(t+t',x)] \\ & - E^*[F(S(t),x)]E^*[F(S(t+t',x)]\}dt', \end{aligned} \tag{1.12}$$

respectively.

Storatonovich's work was based upon the assumption that $X(t)$ does not diverge, in other words, F satisfies a growth condition. Therefore, it is not appropriate manner to apply the Stratonovich's method directly to the randomized Paris' law. In this respect, Tsurui and Ishikawa introduced the absorbing state which corresponds to infinity. They called this state a *death point* and constructed the state space for X in $R^+ \cup \{\infty\}$, where R^+ means a set of positive real number. They defined the conditional mean $E^*[\bullet]$ in the calculation of which the sample paths of X that have reached the death point are removed, and they success to derive the similar Fokker-Planck equation. But it is noticed that the meaning of the cumurative probability is limited in the real space which does not include the death point, that is,

$$\int_{x\in R^+} w(x,t|x_0)dx + \Pr[X \in \{\infty\}] = 1. \tag{1.13}$$

Applying this extended Markov approximation method to the Paris' law, Tsurui and Ishikawa have succeeded in deriving the crack length probability density function and the residual life distribution function in a closed form. The advantages of the Markov approximation method by Tsurui and Ishikawa are listed below [11-13].

(1) Since both spatial and time variables are expressed in continuous space, the mathematical analysis is easy.

(2) Material and statistical parameters are naturally introduced in the stochastic model.

(3) The crack length is expressed as an independent variable in the equation.

(4) It is easy to extend to the mutli-variate problem without any change in the model.

Especially, item (3) is worthy in application to the reliability evaluation in the in-service inspection, and item (4) is suitable for the analysis for random propagation of a surface crack and collinear cracks, which are argued in the later chapters.

For the first time, Tsurui *et al.* applied the above mentioned Markov approximation method to the random crack propagation process under the random loading amplitude [15], and, subsequently, to the process under the random propagation resistance [16]. In 1986, H. Tanaka *et al.* applied the method to the propagating process in which both loading amplitude and propagation resistance are random [17]. Moreover, they extends the Markov approximation method to the random propagation in a body with a finite size [18]. In the present time, the application to the surface crack propagation as two dimensional process is in progress.

In addition to Bogdanoff's theory and Tsurui's theory, the following studies on the random crack propagation are found up to now: Sobczyk proposed an analytic method which transform the randomized Paris' law into a stochastic diffrential equation of Ito type [14], (b) Ditlevsen proposed a randomization technique of the Paris' law in a logarithmic form, and analyzed it with the use of an idea of first passage problem [15].

We now turn to some topics on in-service inspection problem to which the Markov approximation method is applied. The in-service inspection is one of the most important problems on the structural reliability design. The half of the present work is devoted to this theme.

The safety design of the aircraft at present is classified into three policies, that is, the safety life design, the fail-safe design, and the fault tolerant design. The safety life design is a policy to exchange the components or parts which have reached their safety life, which is much shorter than the real life to fail, regardless to defects in the material. The fail-safe design is a policy to prevent the breakdown of the structural system by means of redundant components which supplement the damaged function of the broken component. The fault tolerant design is a policy to keep the amount of damage within a safety level which does not cause the failure. For the purpose, the in-service inspection is repeatedly applied in the service period, and the component is repaired or exchanged if any defects are detected. In order to maintain the reliability by the in-service inspection, it needs to know the probability of the detection (POD). POD means an ability to detect a crack, so it depends on the size of crack at least. The POD curves are researched for the magneto-electric inspection method, ultrasonic inspection method, etc., and they show rather different aspects for each inspection method.

The in-service inspection problems on the structural reliability assurance are argued in the section 2 and 3. Needless to say, the in-service inspection is very important in order to maintain structural components to highly reliabiliable state. If any cracks are detected in the inspection, the components are repaired or exchanged to maintain its reliability. This in-service inspection is usually applied on the aircraft, pressured vessels and piping, etc., which are required for very high reliability. But, on the other hand, an elaborate inspection produces an expensive cost, and too many

inspections bring about the loss of the service from an impatient number of service closing times. Taking account of these safety assurance and economical costs, the inspection method and times must be carefully chosen.

For the theoretical reliability analysis under in-service inspections, it is indispensable to know the distribution of the crack length at arbitrary time because the probability of the crack detection depends, at least, on the crack length. Tsurui's model is preferable to this purpose, since the crack distribution function is expressed in a closed form. In addition, the included parameters are reduced to the material related and external loading related features. Tsurui and Sako [20] was researched the effect of periodical inspection on the structural component, making use of the Markov approximation method for the random crack growth. For the periodical inspection, the crack length distribution at each time can be calculated in advance, so that the transition of the reliability is expressed by the Markov chain. As the result, the necessary inspection times and the inspection period to maintain the required reliability are evaluated under the given condition of the crack detection probability and the crack growth parameters.

In the practical maintenance of the structure, however, the inspections are not always performed periodically. The crack is hardly detected in the early period of the crack growth, and the hazard of the failure increases rapidly in the later period. The inspection periods often become shorter as it is closer to its life. So, it is also necessary to examine the repeated inspection which is not always periodical. We are also interested in the problem when the inspections should be performed so as to optimize the inspection efficiency.

Moreover, we encounter the case that the inspection intervals are changed according to the result of the latest inspection. It seems to be effective to apply a detailed inspection to the special important component, the failure of which causes fatal damage on the structure. If any cracks are detected and the component is exchanged, the structure is safe for a certain period. If not detected, cracks does not exist or are overlooked in the inspection. By the use of the information from the inspection, one can expect less inspection times than those by other methods to maintain the same reliability level. In this work, such inspection policy is called history-dependent inspection policy.

We close this chapter by giving the summary of the contents of the following chapters. In chapter 2, the aperiodic inspection policy which does not depends on the history is investigated in detail. The reliability evaluation method is established in the simple manner and is reduced to three essential equations, which express the initial distribution of the crack length, the crack growth during the inspections, and the crack detection and the repair of the component. Using this method, one can evaluate the reliability of the component at arbitrary time. For further application, the optimal inspection plan can be found under the constriction of inspection times. The content of this chapter was taken from the published paper [21,22].

In chapter 3, the history dependent inspection policy is argued. The reliability analysis for this inspection policy is executed under the assumption that the component in which any crack is detected is exchanged for new component and that the inspection intervals are reset to the same intervals as those for the first component. The mean inspection times as well as the reliability function are calculated with the aid of the idea from the renewal process. There, it is also observed that the randomness of the crack has remarkable influences on both the reliability function and the mean inspection times. This means that the stochastic nature of the crack growth process should be taken into account in order to plan the effective inspection time. The content of this chapter was taken from the published paper [21,23].

In chapter 4, the random propagation of two collinear cracks is investigated. Rivet holes on the aircraft body and wings often make cracks. The plural cracks which has arisen on each holes join each other to cause the instantaneous failure, like zipping off. This type of damage is called Multiple Site Damage (MSD). On the contrary, the damage by one large crack can be called Single Site Damage (SSD). MSD occurs chiefly in the aged aircraft and has some features which are not seen in the single site damage.

(1) The resident strength for the damage by combined cracks is less than that by single crack.

(2) The ductile region of MSD cracks interacts with next cracks and they combine at almost the same time, that is known as unzip effect.

(3) MSD causes the failure with shorter length of crack than SSD. Therefore, the inspection method and intervals for MSD must be decided by the different stand point from SSD.

From these features, it is presumed that the MSD is more sensitive than SSD with respect to the change of the length of each crack, so that the randomization of the crack growth produces considerable effect on the life up to their combination. Of course, it is not suitable to estimate the failure condition by the model on the basis of the independently propagating cracks. The value of the stress intensity factor K under the MSD environment has not been found generally except for a few special conditions. As a simplest example, the combination of two collinear cracks are examined from the

view point of random propagation with bivariate stochastic process. The reliability analysis are performed through the numerical calculation for two dimensional Fokker-Planck equation. The content of this chapter was taken from the published paper [24].

In chapter 5, the random propagation of semi-elliptical surface crack is investigated as bivariate stochastic process. The surface crack often becomes a dangerous factor in piping system or in pressure vessels. The surface crack grows at the different rate in each direction, so that its aspect ratio varies temporally. The type of the failure is classified into two modes. One is that the pressured liquid in the pipe leaks through the hole due to the crack which penetrates the pipe. In this mode, the pipe breaks after the leak with a certain delay time. The other is that the crack length along to the surface exceeds a critical length to cause the instaneneous failure without precedent leak. The former is called Leak Before Break (LBB), and the latter Break Before Leak (BBL). Since the degree of the dangerness is quite different on the above two modes, the failure mode analysis is as important as the life estimation. From this reason, LBB assessment is necessary in the structural reliability analysis of piping system.

If the shape of the crack is elliptic, we describe the propagation of the surface crack with length and depth of the crack. On the reliability analysis for pipes with the surface crack, there exist many uncertain factors such as the number of initial flaws, the initial size of the crack, the physical parameters of the material like a fracture toughness, and so on. This study treats the effect of the random external loading and the random configuration of the material's resistance against the crack propagation. The crack propagation process is regarded as two dimensional stochastic process which corresponds to the length and the depth of the crack. In order to construct the stochastic model, it is assumed that the propagation to each direction is driven by the Paris's law and that the stress intensity factors are described by the Newman-Raju's solution for a semi-elliptical surface crack. Applying the Markov approximation method, two dimensional Fokker-Planck equation is derived. The joint distribution function for crack length and depth is expressed by the solution of the equation. Making an approximation on the propagation process, an analytic solution is given. On the assumption that LBB initiates when the depth of the crack reaches the thickness of the pipe before the length of the crack reaches a critical length, an occurrence rate of LBB can be given in a closed form. Using the occurence rate of LBB, we can get the probability that LBB occurs. The content of this chapter was taken from the published paper [25,26].

In chapter 6, the random crack propagation including overloads is argued. In the metallic material under the repeated loads, it is known that the propagation velocity of fatigue crack becomes slow after arriving of the overload the amplitude of which exceeds the level of normal loading amplitude. This phenomenon is called the retardation effect by overload. This phenomenon implies a difficult problem in the random crack propagation, for the random loads includes some overloads with a certain probability. The propagation model with retardation effect is not found yet, but the retardation time is investigated by several researchers. In order to introduce the retardation effect to the random crack propagation model, H. Tanaka *et al*. [27,28] have modified the residual life distribution function by means of the distribution of retardation time proposed by Arone [29,30], where the repeated loads are divided into the normal loads and overloads. On the other hand, the overload may cause the ductile fracture, depending on the length of the crack, that is, the failure condition at least includes the fatigue mode and the ductile mode, which is often called two–criteria approach. In this study, the two–criteria approach is realized by the Burdekin–Stone's failure condition. The fast Monte–Carlo simulation technique, known as an importance sampling method, is applied to the calculation of the failure probability and the hazard rate of failure. The importance sampling method is frequently utilized in the failure probability evaluation, especially for the multi dimensional state space. The advantage of this method is mentioned by Schuëller et al. [31]. The content of this chapter was taken from the published paper [32].

Chapter 7 is a summary of the present work in chapter 2 to 6. Some remarks and conclusions are made on Markov approximation method for stochastic process in fatigue crack which is applied to the in–service inspection, the bivariate process, and the retardation effect.

References

[1] A. M. Freudenthal, *Trans. of ASCE* **112** (1947), 125–159.

[2] A. M. Freudenthal, J. M. Garrelts and M. Shinozuka, *J. of Structural Div. ASCE* **92** (1966), No. ST1, 267–325.

[3] J. L. Bogdanoff, *Trans. of ASME, J. of Applied Mechanics* **45** (1978), 246–250.

[4] J. L. Bogdanoff and W. Krieger, *Trans. of ASME, J. of Applied Mechanics* **45** (1978), 251–257.

[5] J. L. Bogdanoff, *Trans. of ASME, J. of Applied Mechanics* **45** (1978), 733–799.

[6] J. L. Bogdanoff and F. Kozin, *Trans. of ASME, J. of Applied Mechanics* **47** (1980), 40–44.

[7] F. Kozin and J. L. Bogdanoff, *Eng. Frac. Mech.* **14** (1981), 59–89.

[8] Y. K. Lin and J. N. Yang, *Eng. Frac. Mech.* **18**(1983), 243–256.

[9] P. C. Paris and F. Erdogan, *Trans. of ASME Ser. D*, **85**–4 (1963), 528–534.

[10] R. L. Stratonovich, *Topics in the Theory of Random Noise*, Vol. 1, Chapter 7, London, Gordon and Breach (1963).

[11] A. Tsurui and H. Ishikawa, *Structural Safety* **4** (1986), 15–29.

[12] A. Tsurui and H. Ishikawa, *Trans. of JSME* (in Japanese) A**50** (1984), 31–37.

[13] H. Ishikawa and A. Tsurui, *Trans. of JSME* (in Japanese) A**51** (1985), 1309–1315.

[14] K. Sobczyk, *Eng. Frac. Mech.* **24** (1986), 609–623.

[15] O. Ditlevsen, *Eng. Frac. Mech.* **23** (1986), 467–477.

[16] H. Tanaka, A. Tsurui, and H. Ishikawa, On the cross–effect of uncertainty factors in fatigue crack propagation process, *J. of JSMS* (in Japanese) **35** (1986), 1385–1391.

[17] H. Tanaka and A. Tsurui, Reliability degradation of structural components in the process of fatigue crack propagation under stationary random loading, *Eng. Frac. Mech.* **27** (1987), 501–516.

[18] A. Tsurui, H. Tanaka and T. Tanaka, Probabilistic analysis of fatigue crack propagation in finite size specimens, *Prob. Eng. Mech.* **4** (1989), 120–127.

[19] H. Tanaka, Stochasitc Properties of semi–elliptical surface crack propagation based upon the Newman–Raju's K expression, *Eng. Frac. Mech.* **34** (1989), 189–200.

[20] A. Tsurui and A. Sako, Recent Studies on Structural Safety and Reliability (CJMR, Vol. 5), p.153 (1989), Elsevier.

[21] A. Tsurui, A. Sako and T. Tanaka, Effect of repeated inspections on Structural reliability degradation, *J. of JSMS* (in Japanese) **39** (1990) 748–752.

[22] T. Tanaka and A. Tsurui, Reliability assurance for fatigue crack growth by repeated in–service inspections, *Structural Safety* **9** (1991) 305–314.

[23] A. Tsurui and T. Tanaka, On history–dependent inspection policy for stochastic fatigue crack propagation, *J. of JSMS* (in Japanese) **41** (1992) 1025–1029.

[24] T. Tanaka, A. Yamane A. Tsurui and H. Tanaka, A probabilistic approach to the random propagation of two collinear cracks, *J. of JSMS* (in Japanese) **42** (1993) 1400–1405.

[25] A. Tsurui, T. Tanaka and H. Tanaka, A stochastic fracture mechanical approach to LBB assessment for pipings, *Proceedings of SMiRT 11*, Vol.M (1991) 259–264.

[26] A. Tsurui, H. Tanaka and T. Tanaka, An analytical method for leak before break assessment based upon stochastic fracture mechanics, *Nuclear Engineering and Design* **147** (1994) 171–181.

[27] H. Tanaka, A. Tsurui and T. Isobe, Distribution of fatigue crack propagation life with retardation due to superimposed overloads, *J. of JSMS* **39** (1990), 1539–1544.

[28] H. Tanaka, A. Tsurui and T. Tanaka, Hazard rate of structural components under fatigue environment including random overloads, *Proceedings of the 6th International Conference of Mechanical Behavior of Materials*, Vol.1 (1991), 639–644.

[29] R. Arone, Fatigue crack growth under random overloads superimposed on constant–amplitude cyclic loading, *Eng. Frac. Mech.* **24** (1986), 223–232.
[30] R. Arone, On retardation effects during fatigue crack growth under random overloads, *Eng. Frac. Mech.* **27** (1987), 83–89.
[31] G.I. Schuëller and R. Stix, A critical appraisal of methods to determine failure probabilities, *Structural Safety* **4** (1987), 293–309.
[32] T. Tanaka, Reliability analysis of structural components under fatigue environment including random overloads, *Engineering Fracture Mechanics*, to appear.

Chapter II

Reliability Assurance for Fatigue Crack Growth by Repeated In-Service Inspections

2.1 Introduction

In such an important structure that is repeatedly loaded during the service period as aircraft, it sometimes happens that a fatigue crack propagates from an initial flaw in a certain component to cause the failure of the structure at last. So the reliability of the structure degrades gradually as the time elapses from the start of its service. However, as is well known, it is so difficult to predict precisely the fatigue crack propagation even under the constant loading amplitude, not to mention the random loading. Consequently, in order to evaluate the reliability degradation due to fatigue crack propagation, a treatment as a stochastic process is indispensable, in addition to the uncertainty due to initial flaws. Although the crack growth process does not always have Markov property, it is often approximated to Markov process for its theoretical advantage. As the result, several kinds of stochastic crack growth models are proposed up to now[1-7].

Anyway, it is a matter of course that the important structure should generally be maintained in high reliability. For this purpose, it is a standard practice to detect fatigue cracks by in-service inspections before their size reaches a certain critical length, and to repair or to exchange the components if cracks are detected. It should be noted, however, that too many inspections bring about exceeding decrease of availability and much economical load, and that inspections are constrained with respect to the frequency. Moreover, any inspection is far from perfect, however precise it may be, that is to say, it is not possible to detect all cracks exceeding a specific size. Here we again encounter another uncertainty. At least, the capability of detection depends on the crack length, and discussions on in-service inspections necessarily require an explicit distribution function for the crack length. Hence, it becomes of great interest how the randomness of the process affects the reliability under repeated in-service inspections.

Standing on this point of view, Tsurui and Sako previously studied the effect of periodic inspections on the reliability degradation due to fatigue crack growth under stationary random loading with the aid of a stochastic crack growth model, on the assumption that the component is immediately exchanged if cracks are detected in the inspection[8]. In the engineering reality, however, the inspections are not necessarily performed periodically, and more importantly, the crack growth resistance fluctuates randomly as experiments have pointed out.

In this chapter, therefore, making use of a stochastic crack growth model [9], which includes randomness due to the material inhomogeneity as well as randomness due to loading processes, we discuss the reliability degradation for a model that the component is exchanged when cracks are detected. A method is first presented to evaluate the failure probability under repeated aperiodic inspections. The results are then numerically applied to investigate how the failure probability behaves for a case of ultrasonic inspection method.

As a result, it is clarified that the reliability is significantly affected not only by the number of inspections but also by the policy of inspections or the assignment of the intervals between inspections. In addition, the optimal policy is numerically argued to assign the intervals for a practical example.

2.2 Crack growth model

In this section, we briefly review the stochastic crack growth model by Tsurui and Ishikawa to apply in the later discussion [6]. Some distinctive features of this model run as follows:

- Such characteristic quantities of the stochastic process as a correlation time or length are naturally introduced into the theory.
- Its physical meaning is clarified by the basis of fracture mechanics.
- It is easily applied to the random loading amplitude and/or the random growth resistance.
- Identification of the parameters is not so difficult.
- It is widely applicable to reliability analyses because of the explicit expression of the probability distribution function [6-9].

Here we cite the results from the stochastic crack growth model which includes randomness due to the loading amplitude as well as randomness due to the crack growth

resistance [9].

Now, let Z_n and X be a normalized random stress amplitude at the n-th cycle of loading and a normalized crack length, respectively, then the non-dimensional stress intensity factor range is expressed as $Z_n\sqrt{X}$. Suppose that the Paris-Erdogan's crack propagation law is valid under the random loading and that the crack growth resistance εC_n also fluctuates randomly at every cycle, where ε is a smallness constant of the order of the growth resistance. Then we can utilize

$$\frac{dX}{dn} = \varepsilon C_n\left(Z_n\sqrt{X}\right)^{2(\lambda+1)} \tag{2.1}$$

as a stochastic crack growth model, where $2(\lambda+1)$ is a material constant. Generally, the processes Z_n and C_n do not always have Markov property, but an approximate treatment to the Markov process theory is applicable if the time duration of our main concern is sufficiently large in comparison to the correlation time of the process. According to the Markov approximation study by Tsurui and Ishikawa [6], the cumulative probability distribution function for the crack length X at the n-th cycle of loading under the condition that the initial length was x_0 results in

$$W(x,n|x_0) = \int_0^x w(\xi,n|x_0)d\xi = \Phi\left[\frac{x_0^{-\lambda} - x^{-\lambda} - \lambda\int_0^n \beta(n')dn'}{\lambda\sqrt{\int_0^n \gamma(n')dn'}}\right], \tag{2.2}$$

where $\Phi[\cdot]$ stands for the standard normal distribution function, and

$$\beta(n) = \varepsilon E[C_n Z_n^{2(\lambda+1)}], \tag{2.3}$$

$$\gamma(n) = 2\varepsilon^2\int_{-\infty}^0 \{E[C_n Z_n^{2(\lambda+1)} C_{n+n'} Z_{n+n'}^{2(\lambda+1)}] - E[C_n Z_n^{2(\lambda+1)}]E[C_{n+n'} Z_{n+n'}^{2(\lambda+1)}]\}dn'. \tag{2.4}$$

$w(x,n|x_0)$ is interpreted as a transition probability density function with respect to X, but it is noted that it does not satisfy the normalization condition in the usual sense, that it to say,

$$\lim_{x\to\infty} W(x,n|x_0) = \int_0^\infty w(x,n|x_0)dx \le 1. \tag{2.5}$$

This apparent anomaly of the probability is an inevitable problem on the stochastic discussion of the crack propagation, since the Paris-Erdogan's law (2.1) does not satisfy the growth condition for the existence of its stochastic solution if $\lambda>0$, i.e., X is explosively increasing and may possibly escape to infinity in any cycle. Therefore, we must introduce such an absorbing state corresponding to infinity as a death point, in which X no longer obeys the growth eqation. This mathematical difficulty can be avoided by means of extending the interpretation of the Fokker-Planck equation derived from the growth equation, to the process with this growth singularity through some complicated mathematical operations. As usual, $W(x,n|x_0)$ can be interpreted as a probability that the crack length X is equal to or less than x at the n-th cycle and $1-W(x,n|x_0)$ as a probability that X is greater than x or lies in the death point.

Now, assume that Z_n and C_n are statistically independent and both locally stationary and that the correlation of each process decays exponentially. Introducing a new variable and new statistical parameters as

$$t = \varepsilon E[C_n]E[Z_n^{2(\lambda+1)}]n, \tag{2.6}$$

$$\zeta_z^2 = \frac{E[Z_n^{4(\lambda+1)}]-\left(E[Z_n^{2(\lambda+1)}]\right)^2}{\left(E[Z_n^{2(\lambda+1)}]\right)^2}, \tag{2.7}$$

$$\zeta_c^2 = \frac{E[C_n^2]-\left(E[C_n]\right)^2}{\left(E[C_n]\right)^2}, \tag{2.8}$$

$$t_z = \frac{2\varepsilon E[C_n]E[Z_n^{2(\lambda+1)}]}{E[Z_n^{4(\lambda+1)}]-\left(E[Z_n^{2(\lambda+1)}]\right)^2}\int_{-\infty}^{0}\left\{E[Z_n^{2(\lambda+1)}Z_{n+n'}^{2(\lambda+1)}] - E[Z_n^{2(\lambda+1)}]E[Z_{n+n'}^{2(\lambda+1)}]\right\}dn', \tag{2.9}$$

we can rewrite eqn. (2) as the following form [8]:

$$W(x,n|x_0) = \int_0^x w(\xi,n|x_0)d\xi = \Phi\left[\frac{x_0^{-\lambda}-x^{-\lambda}-\lambda t}{\lambda\sqrt{G_1(t)+G_2(t)+G_3(t)}}\right], \tag{2.10}$$

where,

$$G_1(t) = \begin{cases} \dfrac{\alpha_0\zeta_c^2}{2\lambda+1}\left\{x_0^{-(2\lambda+1)} - (x_0^{-\lambda}-\lambda t)^{(2\lambda+1)/\lambda}\right\}, & \text{for } t < \dfrac{x_0^{-\lambda}}{\lambda}, \\ \dfrac{\alpha_0\zeta_c^2}{2\lambda+1}x_0^{-(2\lambda+1)}, & \text{for } t \geq \dfrac{x_0^{-\lambda}}{\lambda}, \end{cases} \tag{2.11}$$

$$G_2(t) = \zeta_z^2 t_z t, \tag{2.12}$$

$$G_3(t) = \begin{cases} \zeta_z^2\zeta_c^2 t_z\displaystyle\int_0^t \frac{dt'}{1+(t_z/\alpha_0)(x_0^{-\lambda}-\lambda t')^{-(\lambda+1)/\lambda}}, & \text{for } t < \dfrac{x_0^{-\lambda}}{\lambda}, \\ \zeta_z^2\zeta_c^2 t_z\displaystyle\int_0^{x_0^{-\lambda}} \frac{dt'}{1+(t_z/\alpha_0)(x_0^{-\lambda}-\lambda t')^{-(\lambda+1)/\lambda}}, & \text{for } t \geq \dfrac{x_0^{-\lambda}}{\lambda}. \end{cases} \tag{2.13}$$

These uncertainty factors are, respectively, caused by the randomness of the crack growth resistance, the randomness of the loading amplitude and the cross effect of both processes. Note that G_1 and G_3 depend not only on t but also on the initial crack length x_0. The variable t will be referred by the term "time", which is used instead of loading cycle n hereafter. The parameters ζ_z and ζ_c denote the coefficient of variation of $Z_n^{2(\lambda+1)}$ and C_n, respectively. t_z means a correlation time of $Z_n^{2(\lambda+1)}$ scaled in the same way as t, and α_0 is an unknown material constant.

It is worth noting that the result above comes from the following considerations. Essentially, C_n needs to be stationary with respect not to n but to the increment of the crack growth length, for material inhomogeneity is supposed to distribute uniformly with

a spatial correlation. Therefore, more natural is the idea that the correlation cycle for C_n should decrease as the crack growth rate increases. Since the growth rate is also a stochastic quantity, the strict discussion will be very difficult, but, as a first approximation, we can make an assumption that the correlation cycle n_c of the growth resistance should be inversely proportional to the conditional mean of the crack growth rate under the condition that X does not lie in the death point,

$$n_c \approx \alpha_0 / \tilde{E}\left[\frac{dX}{dn}\right], \tag{2.14}$$

where the constant α_0 corresponds to the spatial correlation length of the growth resistance.

2.3 Reliability evaluation method under repeated inspections

In this section, we will develop a method to evaluate the reliability of a component under repeated inspections, the plan of which is pre-specified in advance of the service and never modified throughout the service period by any information from the results of inspections. Here we concern ourselves with a simple repair method that the defective component is immediately exchanged for a new one if cracks are detected through the inspections.

Let t_j be the time interval between $(j-1)$-th and j-th inspections. We begin with the classification of events in which the state of component lies at the j-th inspection time. $\boldsymbol{F}_j$ is introduced to express the event that the failure occurred by the j-th inspection time, and $\boldsymbol{S}_j$ the event that the failure did not occur from the start of the operation up to the j-th inspection time. Further, $\boldsymbol{S}_j$ is separated by the result of the j-th inspection into two events, which are $\boldsymbol{D}_j$ that at least one crack is detected in the component and $\boldsymbol{U}_j$ that no cracks are detected. Thence, the three states of the component at the moment just after the j-th inspection $\boldsymbol{F}_j$, $\boldsymbol{S}_j \cap \boldsymbol{D}_j$ and $\boldsymbol{S}_j \cap \boldsymbol{U}_j$ constitute a disjoint and exhaustive system of events. Consequently, their outcome probabilities enjoy the relationship

$$P_F^{(j)} + P_D^{(j)} + P_U^{(j)} = 1 \tag{2.15}$$

for arbitrary j, where,

$$P_F^{(j)} = \Pr[\boldsymbol{F}_j],\ P_D^{(j)} = \Pr[\boldsymbol{S}_j \cap \boldsymbol{D}_j],\ P_U^{(j)} = \Pr[\boldsymbol{S}_j \cap \boldsymbol{U}_j]. \tag{2.16}$$

Note that $P_D^{(j)}$ and $P_U^{(j)}$ represent the j-th detection and undetection probability, respectively, without the condition that the component has not failed yet by the j-th inspection time, and that $P_F^{(j)}$ means the failure probability up to the j-th inspection time. Since failed components are excluded from the service after the failure occurred, the failure state F_j becomes an absorbing state. The components only in the state $S_j \cap D_j$ are repaired to be exchanged, and those in $S_j \cap U_j$ are left intact.

Suppose that $\tilde{u}_j(x)$ and $u_j(x)$ mean crack length density functions just before and just after the j-th inspection, respectively, on which the same normalization condition

$$\int_0^{x_c} \tilde{u}_j(x)dx = \int_0^{x_c} u_j(x)dx = 1 - P_F^{(j)} \tag{2.17}$$

is imposed, where x_c stands for a pre-specified critical crack length beyond which the structural component fails. Note that these density functions are not conditional on the state of the component, but consistent with transition probability density function $w(x,t|x_0)$ introduced in the preceding section.

When the j-th inspection is performed, the crack length density function can be expressed as

$$\tilde{u}_j(x) = P_D^{(j)}\tilde{u}_j(x|S_j \cap D_j) + P_U^{(j)}\tilde{u}_j(x|S_j \cap U_j). \tag{2.18}$$

Here $\tilde{u}_j(x|\cdot)$ stands for the conditional crack length density function. If $g(x)$ means the initial crack length density function of new components, then to exchange the crack detected component for the new one is nothing but to replace $\tilde{u}_j(x|S_j \cap D_j)$ with $g(x)$. Therefore, the following relationship holds between the densities just before and just after the j-th inspection time:

$$u_j(x) = \begin{cases} g(x) & \text{for } j = 0, \\ P_D^{(j)}g(x) + P_U^{(j)}\tilde{u}_j(x|S_j \cap U_j) & \text{for } j > 0. \end{cases} \tag{2.19}$$

It is noted that $u_j(x)$ does not depend on the history of the maintenance, and that it is a kind of virtual distribution function rather than the actual one which will be found at each inspection time.

Further, if $D(x)$ means a probability of detection for a crack of size x, to which we will refer as a detectability of the crack, then Bayes's formula leads to

$$P_U^{(j)}\tilde{u}_j(x|S_j \cap U_j) = \{1 - D(x)\}\tilde{u}_j(x). \tag{2.20}$$

Therefore, making use of the relation

$$P_D^{(j)} = \int_0^{x_c} D(x)\tilde{u}_j(x)dx, \tag{2.21}$$

we can interpret eqn.(2.19) as a bridge between $\tilde{u}_j(x)$ and $u_j(x)$.

On the other hand, in accordance with the crack propagation from the $(j-1)$-th to the j-th inspection time, $\tilde{u}_j(x)$ can be easily connected with $u_{j-1}(x)$ as

$$\tilde{u}_j(x) = \int_0^{x_c} w(x,t|x_0)u_{j-1}(x_0)dx_0 \quad (j > 0), \tag{2.22}$$

from the assumed Markov property.

For summary, the crack length density functions $\tilde{u}_j(x)$ and $u_j(x)$ ($j = 0,1,2,\ldots$) at each inspection time can be recursively calculated through the following three equations.

- Initial density:

$$u_0(x) = g(x). \tag{2.23}$$

- Crack propagation:

$$\tilde{u}_j(x) = \int_0^{x_c} w(x,t|x_0)u_{j-1}(x_0)dx_0 \quad (j > 0). \tag{2.24}$$

- Inspection and repair:

$$u_j(x) = P_D^{(j)}g(x) + \{1 - D(x)\}\tilde{u}_j(x). \tag{2.25}$$

Now, with the aid of these functions, we will calculate the cumulative failure probability $H(t)$ of the component up to time t. For convenience sake, we will introduce the time interval $s_k = t_1 + t_2 + \cdots + t_k$, $(s_0 = 0)$. For the interval $s_k < t \le s_{k+1}$, as the difference between $H(t)$ and $P_F^{(k)}$ is nothing but the probability that the component fails in the time interval (s_k, t),

$$\begin{aligned} H(t) &= 1 - \int_0^{x_c} W(x_c, t - s_k|x_0)u_k(x_0)dx_0 \\ &= P_F^{(k)} + \int_0^{x_c} \{1 - W(x_c, t - s_k|x_0)\}u_k(x_0)dx_0. \end{aligned} \tag{2.26}$$

Especially, putting $t = s_{k+1}$, we get the expression for the failure probability up to $(k+1)$-th inspection time as

$$\begin{aligned} P_F^{(k+1)} &= 1 - \int_0^{x_c} \tilde{u}_{k+1}(x_0)dx_0 \\ &= P_F^{(k)} + \int_0^{x_c} \{1 - W(x_c, t_{k+1}|x_0)\}u_k(x_0)dx_0. \end{aligned} \tag{2.27}$$

It is worth noting that one had better make use of the first expressions of eqns. (2.26) and (2.27) to give accurate numerical results in the range where $H(t)$ or $P_F^{(k)}$ is greater than 0.5, and the second expressions otherwise.

2.4. Numerical examples

Here, we will give an example to show the feasibility of the method proposed in the preceding section. For this purpose, it is required to specify the initial crack length density $g(x)$ and the crack detectability $D(x)$. Now, assuming that the time T_0 when the crack size reach a specific length $\tilde{x}$ obeys a Weibull distribution with two parameters [10], and that the initial crack length X_0 is related to T_0 through $T_0 = c(X_0^{-\lambda} - \tilde{x}^{-\lambda})$ $(0 < X_0 < \tilde{x})$, then we will fix a probability density function $g(x)$ of the initial crack length X_0 through

$$\int_0^x g(x)dx + \Pr[T_0 \le t_0] = 1, \tag{2.28}$$

where,

$$\Pr[T_0 \le t_0] = 1 - \exp\left\{-\left(\frac{t_0}{\beta}\right)^{\gamma}\right\}. \tag{2.29}$$

Figure 2.1 shows an example of this density function $g(x)$, which will be made use of in what follows ($\lambda = 0.5$, $\tilde{x} = 0.6$, $c/\beta = 1.63$, $\gamma = 4.0$).

On the other hand, according to Harris and Lim [11], we will introduce a crack detectability function for ultrasonic inspection technique as

$$D(x) = 1 - \frac{1}{2}\,\mathrm{erfc}\left\{\alpha \ln\left(\frac{x}{x^*}\right)\right\}, \tag{2.30}$$

where α and x^* are adequate parameters. Figure 2.2 corresponds to typical one ($\alpha = 2.1$, $x^* = 1.0$).

Now, assuming the distribution of initial flaws and the detectability of cracks as shown in figs. 2.1 and 2.2, we will observe the behavior of the failure probability $H(t)$. Here, the time variable following figures of the failure probability curves.

Figure 2.3 shows the temporal variation curves of the failure probability of the component. The dotted curve corresponds to a case in which we perform no inspections. We can see from this curve that the component fails certainly by $t = 5.0$ unless inspections are performed. The solid curves A, B and C correspond to the cases in which we perform three times of inspections by $t = 5.0$ under the following three plans,

A: $t_1 = t_2 = t_3 = t_4 = 1.25$;

B: $t_1 = 1.66$, $t_2 = t_3 = 1.17$, $t_4 = 1.0$;

C: $t_1 = 2.0$, $t_2 = t_3 = t_4 = 1.0$.

The parameters are set as $\lambda = 0.5$, $t_z = 0.02$, $\alpha_0 = 0.32$, $\zeta_z = 0$, $\zeta_c = 0.3$, and $x_c = 231$. ($\zeta_z = 0$ means that the loading amplitude is deterministic.) It is recognized that the inspection policy has a significant effect on the reliability curves. The first long intervals raises the final reliability in this example, because of no possibility of critical cracks in new components and of good detectability of large cracks.

Figures 2.4 and 2.5 show $H(t)$ curves for $\zeta_z = 0$, $\zeta_c = 0$ (the loading amplitude and the growth resistance are both deterministic), and for $\zeta_z = 1.0$, $\zeta_c = 0.3$ (both random), respectively, under the same inspection plans as in fig. 2.3, keeping the other parameters unchanged. These curves on the figures show us that the randomness of propagation resistance considerably affects on the reliability degradation as well as the inspection policy does. We cannot assert, however, that the large uncertainty factor should always let down the reliability if the inspections are frequently repeated. The larger variance of crack length may produce the greater probability of detection and exchange, and, as the result, may possibly raise the total reliability.

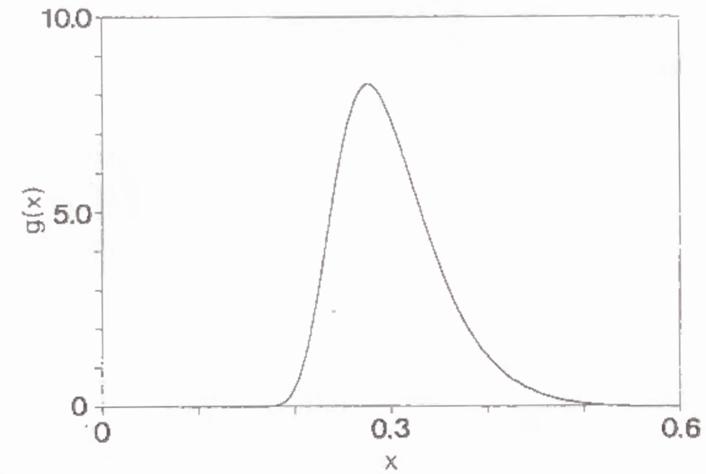

Fig. 2.1. The probability density function of the initial crack length.

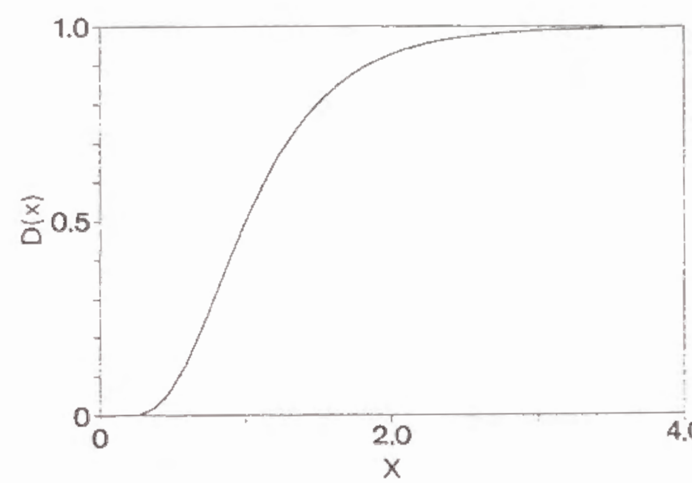

Fig. 2.2. The probability of detection probability as a function of crack length.

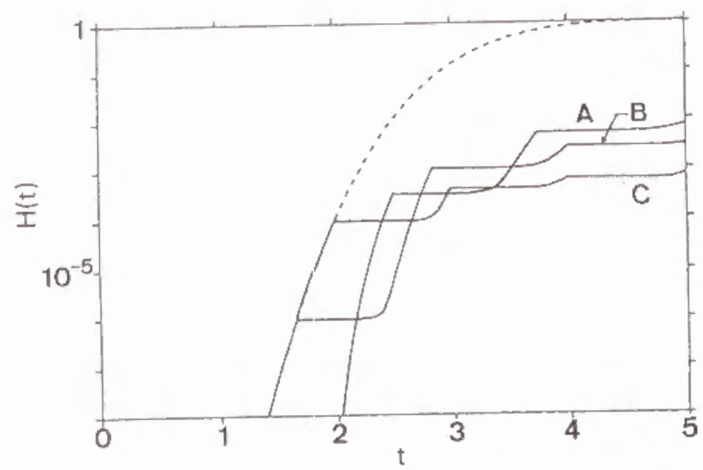

Fig. 2.3. The failure probability under repeated inspections ($\zeta_z = 0.0$, $\zeta_c = 0.3$).

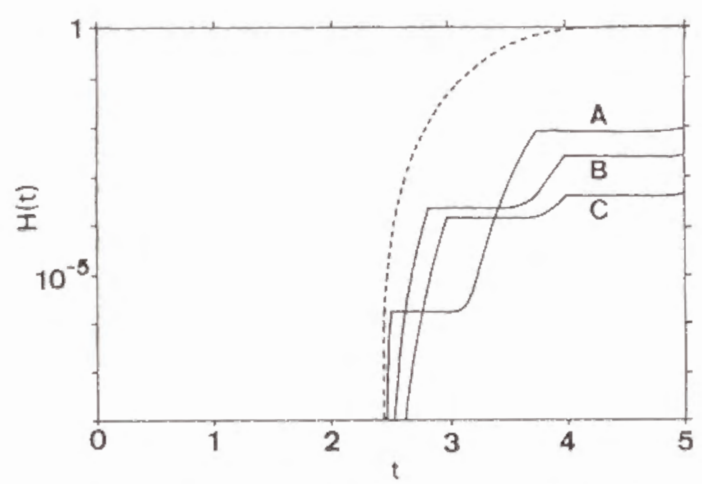

Fig. 2.4. The failure probability under repeated inspections ($\zeta_z = 0.0$, $\zeta_c = 0.0$).

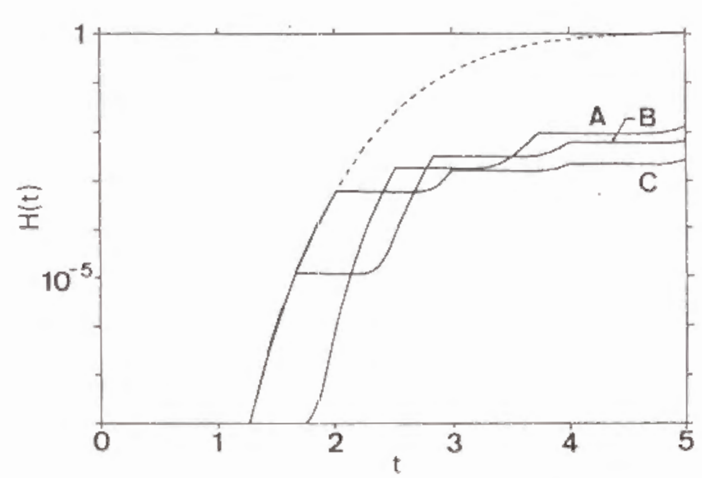

Fig. 2.5. The failure probability under repeated inspections ($\zeta_z = 1.0$, $\zeta_c = 0.3$).

Usually, such a detailed inspection cannot be carried out so many times owing to the economical cost, which often imposes some constraints on the frequency of inspections during the service period. In relation to the question how many inspections are required in order to maintain the necessary reliability level, it is a matter of interest to what level the reliability can be raised within a definite number of inspections by the choice of the inspection plan.

Now, let T be a total service period and $H(t_1, t_2, \cdots, t_{n+1})$ denote $P_F^{(n+1)}$ in eqn. (2.27) as a function of inspection intervals, i.e., the failure probability for n times of inspections. The problem is to decide the intervals $\{t_j\}$ to minimize $H(t_1, t_2, \cdots, t_{n+1})$ subject to $t_1 + t_2 + \cdots + t_{n+1} = T$. This is equivalent to the problem to maximize $T = t_1 + t_2 + \cdots + t_{n+1}$ subject to $H(t_1, t_2, \cdots, t_{n+1}) = P_F$ (constant).

Since H is given by the complicated form of repeated integrals, it is difficult to solve this optimization problem by analytic manners, generally. But a numerical solution is not impossible for small n. Figure 2.6 shows the minimum value of failure probability H as a function of T, which is directly computed by means of the successive iteration with respect to t_j's.

The curve for $n = 0$ in fig. 2.6 means the failure probability without any inspection, which gives upper bounds of $H(t)$ for all kinds of inspection plans. The curves for $n = 1$ and 2 give lower bounds of the failure probability curves for n times of inspections with fixed total service period. It is of great interest that the condition $t_1 > t_2 > t_3 > \cdots$ does not always give the optimum plan (see Table 2.1).

Of course, the failure probability curve is strongly influenced by the statistical parameters of the crack growth, of the initial flaw and of the detectability, so that the parameter sensitivity analysis will be required for the practical purposes.

2.5 Conclusions

We have investigated the effect of repeated in-service inspections on the reliability degradation due to fatigue crack growth based upon a stochastic fracture mechanics, in consideration of the random loading stress and the random growth resistance simultaneously. A reliability evaluation method for a component exchange model has been proposed and applied to investigate how the reliability of structural component behaves under repeated ultrasonic inspections. Consequently, we saw that the diffusive effect associated with the crack growth processes in addition to the inspection policy has a significant influence on the reliability of the component.

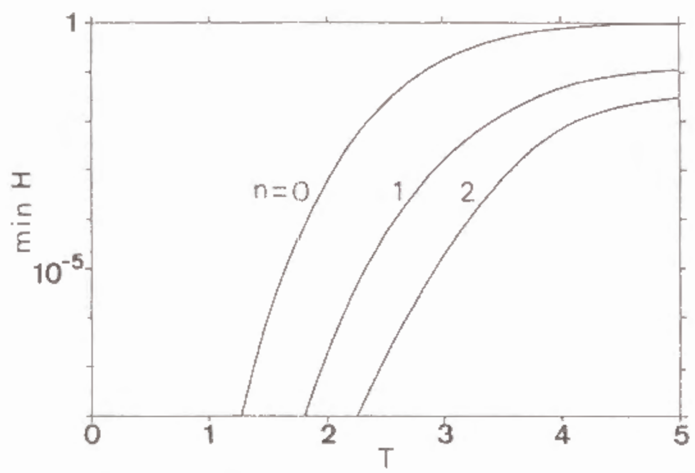

Fig. 2.6. The minimum failure probability at the end of service for n-times of inspections ($\zeta_z = 1.0$, $\zeta_c = 0.3$).

Table 2.1

The optimal inspection intervals for $n = 2$

T	t_1	t_2	t_3
2.5	1.34	0.62	0.54
3.0	1.59	0.74	0.67
4.0	1.99	0.97	1.04
5.0	2.33	1.22	1.45

Moreover, the optimum policy has been numerically argued to assign intervals between the inspections.

It is suggested that one should design the inspection plan based on the understanding of the crack propagation as a stochastic process in addition to the knowledge on the initial flaws of the component and on the detectability of the cracks.

References

[1] J.L. Bogdanoff, A new cumulative damage model, *J. Appl. Mech., ASME* **45** (2) (1978), 246-250.

[2] F. Kozin and J.L. Bogdanoff, A critical analysis of some probabilistic models of fatigue crack growth, *Eng. Fract. Mech.*, **14** (1981) 59-89.

[3] Y.K. Lin and J.N. Yang, On statistical moments of fatigue crack propagation, *Eng. Fract. Mech.*, **18** (1983) 243-256.

[4] O. Ditlevsen, Random fatigue crack growth — A first passage problem, *Eng. Fract. Mech.*, **23** (2) (1986) 467-477.

[5] K. Sobczyk, Modeling of random fatigue crack growth, *Eng. Fract. Mech.*, **24** (4) (1986) 609-623.

[6] A. Tsurui and H. Ishikawa, Application of the Fokker-Planck Equation to a stochastic fatigue crack growth model, *Structural Safety*, 4 (1) (1986) 15-29.

[7] H. Ishikawa, A. Tsurui and H. Kimura, Stochastic fatigue crack growth model and its wide applicability in reliability-based design, *Current Japanese Materials Research*, Vol.**2**, Elsevier, London and New York, 1987, pp.45-58.

[8] A. Tsurui and A. Sako, Reliability analysis of fatigue crack growth processes under stationary random loading, *Current Japanese Materials Research*, Vol.5, Elsevier, London and New York, 1989, pp.153-165.

[9] H. tanaka and A. Tsurui, Reliability degradation of structural components in the process of fatigue crack propagation under stationary random loading, *Eng. Fract. Mech.*, **27** (5) (1987) 501-516.

[10] M. Shinozuka, Development of reliability-based aircraft safety criteria: An impact analysis, *Technical Report AFFDL-TR-76-36*, Vol. 1, Air Force Flight Dynamics Laboratory, Ohio, 1976.

[11] D.O. Harris ad E.Y. Lim, Application of a probabilistic fracture mechanics model to the influence of in-service inspection on structural reliability, *ASTM STP* **798** (1983) 19-41.

Chapter III

On History-Dependent Inspection Policy for Stochastic Fatigue Crack Propagation

3.1 Introduction

In order to maintain the reliability of very important machine or structure under fatigue environment, it is usual manner to perform in-service inspection and to repair or exchange the damaged components if cracks are detected. But, usually, the in-service inspection needs the great cost and the service is not available during the inspection time. For these reasons, it is not recommended to repeat the inspection exceeding many times, so some constraints exists on the number of inspections. Therefore it is an very important study to search the most effective inspection plan and to investigate the reliability degradation under a certain policy of the inspection. Many researches that depends on several crack growth models are appeared up to now.

In this point of view, Tsurui *et al.* have investigated the effects of inspection using the crack growth model based on the stochastic fracture mechanics, which imply the uncertainty on the crack propagation process caused by the random loads and random propagation resistance, in addition to the uncertainty with respect to the detection of cracks or initial crack length. As the result, it is clarified that the correlation time of the loading amplitude and the correlation distance have a great effect on the random crack propagation, and that the treatments as a stochastic process are indispensable for the random loading and propagation resistance. If we do not take these random factor into account, we will estimate the failure probability too small value.

In the preceeding chapter, I researched the effects of repeated in-service inspections, the intervals of which are predetermined and do not depends on the history of the inspection results. However, if we confront the situation that there exists a special important component in the structural system, it may be more better inspection plan that depends on the history of the special important component. So we take notice of a special important components, and consider a certain representative inspection policy that depends the inspection history. A reliability estimation method is developed in this chapter on the assumption that once cracks are detected in the component, it is exchanged for new component, and examined on the utility.

Firstly, the conditional reliability function under the condition that the component was not exchanged up to the time is derived in aid of the density function of the crack length that is derived from Markov approximation method proposed by Tsurui *et al.* Secondly, a kind of functional equation for the reliability function is derived by means of the idea of reproducing process. Last, solving the equation by numerical method, it is clarified how the history-dependent inspection policy contribute to maintaining the reliability of the component. In addition, we derive a similar functional equation for the mean inspection times under this policy, and solve it numerically to evaluate the utility of the inspection policy. As the result, it is known that the difference of the inspection policy have a great influence on the reliability degradation of the components. Especially, it is also known the first inspection interval has a tendency to influence the final reliability level.

3.2 Stochastic crack propagation model

In this section, we briefly review the stochastic crack propagation model, which is used in the later section. This model is a type of Markov approximation method developed by Tsurui *et al.* as a diffusive-type crack propagation model, related in the previous section. For the brief notation, suppose the stress amplitude Z_0 is non-dimensional and constant, that is, the only propagation resistance at the crack tip fluctuates randomly. Now, assume that X is a non-dimensional crack length at n-th cycle of loading, the application of well-known Paris-Erdogan's low for the random propagation resistance is expressed as

$$\frac{dX}{dn} = \varepsilon C_n\left(Z_0\sqrt{X}\right)^{2(\lambda+1)}, \tag{3.1}$$

where $2(\lambda+1)$ is a material constant that correspond to the power index of Paris-

Erdogan's low, εC_n is a random propagation resistance at n-th loading cycle occurred from material inhomogeneity, and ε is a smallness parameter.

According to above-mentioned Markov approximation method, from eqn. (3.1) under appropriate assumption, we get the following the probability distribution function of the crack length $W(x,t|x_0)$ and the probability density function of the crack length $w(x,t|x_0)$,

$$W(x,t|x_0) = \int_0^x w(x,t|x_0)dx = \Phi\left[\frac{x_0^{-\lambda} - x^{-\lambda} - \lambda t}{\lambda\sqrt{G(t)}}\right], \tag{3.2}$$

under the condition that the initial crack length is x_0. In this expression, the new variable

$$t = \varepsilon E[C_n]Z_0^{2(\lambda+1)}n \tag{3.3}$$

is introduced instead of the loading cycle n. The notation $\Phi[\cdot]$ means the standard normal distribution function, and $G(t)$, the uncertainty factor in connection with the dispersion of the crack length, and is expressed as

$$G(t) = \begin{cases} \dfrac{\alpha_0\zeta_C^{\,2}}{2\lambda+1}\left\{x_0^{-2(\lambda+1)} - \left(x_0^{-\lambda} - \lambda t\right)^{(2\lambda+1)/\lambda}\right\} & \text{for } t < \dfrac{x_0^{-\lambda}}{\lambda}, \\ \dfrac{\alpha_0\zeta_C^{\,2}}{2\lambda+1}x_0^{-2(\lambda+1)} & \text{for } t \geq \dfrac{x_0^{-\lambda}}{\lambda}, \end{cases} \tag{3.4}$$

where α_0 is a parameter in connection of correlation distance of the crack propagation resistance, $\zeta_C^{\,2}$, a square coefficient of variation of the random variable C_n defined with

$$\zeta_C^{\,2} = \frac{E[C_n^2] - (E[C_n])^2}{(E[C_n])^2}. \tag{3.5}$$

Note that $w(x,t|x_0)$, the probability density function of the crack length, does not always satisfy the normalization condition.

3.3 History-dependent inspection policy

3.3.1 Inspection policy

In this section, the history-dependent inspection policy is modeled by means the most important structural component in the whole structural system. At the first time, the inspection is performed at t_1 after the structure is started in service. If any crack is detected in the component on the inspection, then the component is exchanged with a new component and the next inspection time is set at t_1 later, otherwise, the component is left intact until the next inspection time t_2. Next, if no cracks are detected in the component in which cracks were not detected at the first inspection, then the component is still kept in service and inspected after t_3 later, otherwise, the components is exchanged and inspected after t_1 later. Similarly, at the n-th inspection time, if no cracks are detected in the component which passed the all inspections before $(n-1)$-th times, then the next inspection is performed at t_n later, otherwise, the component is exchanged and the next inspection time is set at t_1 later, that is, the inspection schedule is reset and returned to the beginning. Figure 3.1 shows the schematic description. In the figure, the notation F, UD and XC means the event that the component failed in the latest interval, the event that no cracks are detected and kept in service, and the event that cracks are detected and the component is exchanged, respectively.

This inspection policy has some complexity in the point that future inspection time is unsettled except the next inspection time, but it is seemed to be more fancy in the point that the expected inspection times are less than those of fixed inspection plan that brings the same in-service reliability.

3.3.2 Reliability estimation method

Now, let $R(t)$ be a reliability function at the time t. In this subsection, we construct a functional equation for $R(t)$ in the domain $s_{n-1} < t \le s_n$, where $s_k = t_1 + t_2 + \cdots + t_k$ $(s_0 = 0)$. By means of probability conditioning at the first exchange of the component, we firstly derive the conditional reliability function $f(t)$ under the condition that the crack detection, that is, the component exchange has never occurred up to the current time.

Let $g(x)$ be a probability density function of the initial crack length. We define functions $v_k(x)$ recursively as

$$v_k(x) = \begin{cases} g(x), & (k=0), \\ \{1 - D(x)\} \int_0^{x_c} w(x, t_k | x_0) v_{k-1}(x_0) dx_0, & (k > 0), \end{cases} \tag{3.6}$$

which are proportional to the probability density function of the crack length just after the each inspection time s_k under the same condition of $f(t)$. In the above expression, $w(x, t_k | x_0)$ is the probability density function of the crack length expressed in the equation (3.2), $D(x)$ the probability of detection (POD) for the crack of length x, the upper bound of the integral x_c the critical crack length. Next, we clarify what normalization condition is imposed on the function sequence $v_k(x)$ $(k = 0, 1, 2, \ldots)$. For $k = 0$, evidently, $v_0(x)$ is normalized to unity in the domain $(0, x_c)$. For $k = 1$, the value of

$$\int_0^{x_c} w(x, t_1 | x_0) v_0(x_0) dx_0$$

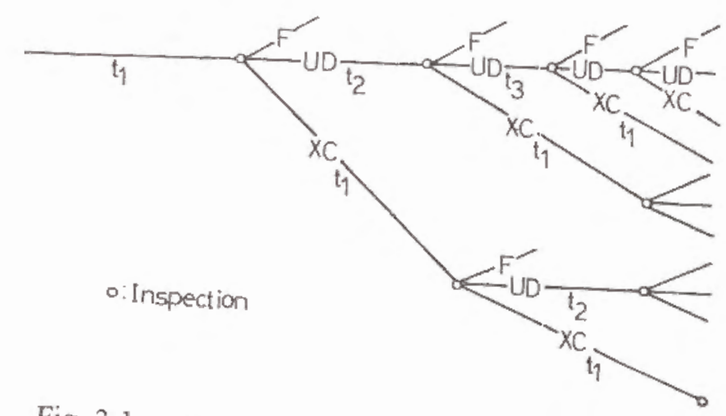

Fig. 3.1. The schematic representation of state transitions for a state-dependent inspection policy.

means the probability that the crack length x is implied by $(0, x_c)$ at the time t_1, that is to say, the probability density function which is normalized to the value of the probability that the component does not failed until the time t_1. Since the probability that the existing crack of length x is not detected is evaluated as $1 - D(x)$, so

$$\{1 - D(x)\}\int_0^{x_c} w(x, t_1 | x_0) v_0(x_0) dx_0$$

is proportional to the probability density function of the crack length just after the inspection time t_1, and is normalized to the probability that the component does not failed and that the crack is not detected in the inspection. The same as above consideration, we see that $v_k(x)$ is normalized to the probability that the component does not failed and that cracks were not detected in the all past inspections at $s_1, s_2, \cdots, s_k$, that is, the exchange of the component has never occurred until the k-th inspection.

Making use of the function sequence $v_k(x)$, it is evident that the restricted functions of $f(t)$ on $s_k < t \le s_{k+1}$ (k = 0, 1, 2,...) are given by

$$f_k(t) = \int_0^{x_c} W(x_c, t - s_k | x_0) v_k(x_0) dx_0 . \tag{3.7}$$

Therefore, for arbitrary n, the value of $f(t)$ in the domain $0 < t \le s_n$ is evaluated by the following equation:

$$f(t) = \begin{cases} f_0(t) & (0 < t \le s_1) \\ f_1(t) & (s_1 < t \le s_2) \\ \vdots & \vdots \\ f_{n-1}(t) & (s_{n-1} < t \le s_n) \end{cases} \tag{3.8}$$

Suppose that the first exchange of the component was occurred at s_k. Then, the occurrence probability, P_k, of this event has a relation with $f(t)$ through

$$P_k = f(s_k) - \int_0^{x_c} v_k(x) dx, \tag{3.9}$$

because the occurrence of the first exchange at s_k is the difference event of detecting no cracks in the all past inspections at $s_1, s_2, \cdots, s_{k-1}$ from the event of no failure and detecting no cracks in the inspection at s_k. Once the component is exchanged for a new component, it traces the same probability transition pattern as the first one, with delay to s_k. So the conditional reliability function at the time t is equal to $R(t - s_k)$.

Now, for $s_{n-1} < t \le s_n$, we get the recursive expression of $R(t)$ from the classification by the first exchange time, that is, the event that no exchanges occurred, that the first exchange occurred at s_1, occurred at s_2, ... , and occurred at s_{n-1}. For n = 2, 3, ..., the expression of $R(t)$ follows:

$$R(t) = f(t) + \sum_{k=1}^{n-1} P_k R(t - s_k), \quad (s_{n-1} < t \le s_n) \tag{3.10}$$

especially, for n = 1, that is, $0 < t \le s_1$, $R(t) = f(t) = f_0(t)$. In order to evaluate $R(t)$ in $s_{n-1} < t \le s_n$, we can perform the backward calculation of $R(t)$ in $s_{n-2} < t \le s_{n-1}$ recursively, and repeat until t drops in $0 < t \le s_1$. Formally, above expression can be extended to $0 < t \le s_n$ if we set $R(t) = 0$ for $t < 0$.

3.3.3 Estimation method of mean inspection times

Also in this subsection, as well as preceding, we consider the time t such that $s_{n-1} < t \le s_n$ and compose a functional equation which the mean inspection time $m(t)$ satisfies. Since the inspection on failed components is nonsense, the number of inspection times after the failure is regarded as zero.

First, we calculate the mean inspection time in case that exchange of the component has never occurred. Recollect that $f(s_k)$ in the preceding subsection stands for the probability that the component is neither failed nor exchanged. In case of no-exchange, the simple summation of $f(s_k)$ from k = 1 to $n - 1$ is nothing else but a conditional mean inspection times at t.

In the next, suppose that the component did not fail until the k-th inspection and that the first exchange has occurred at the k-th inspection. By the same consideration as preceding, the conditional mean inspection times is also equal to $m(t - s_k)$ because of the recurrence of the inspection intervals. Since P_k are the occurrence probabilities of the event that the component did not fail until s_k and that the first exchange has occurred at s_k, the production of both, that is, $P_k m(t - s_k)$ is considered as the classified mean inspection times.

As the result of conditioning with respect to the first exchange time, for n = 2, 3, ... , we get the following equation:

$$m(t) = \sum_{k=1}^{n-1} \{f(s_k) + P_k m(t - s_k)\}, \qquad (s_{n-1} < t \le s_n) \tag{3.11}$$

Evidently, $m(t) = 0$, for n = 1, that is, $0 < t \le s_1$. We are able to judge whether the inspection policy is efficient or not, if we solve the above equation numerically by the iteration. To describe the above equation in the domain $0 < t \le s_n$, the Heaviside function

$$\Theta(t) = \begin{cases} 0, & (t \le 0) \\ 1, & (t > 0) \end{cases} \tag{3.12}$$

are introduced. Then the equation (3.11) is expressed as

$$m(t) = \sum_{k=1}^{n-1} f(s_k)\Theta(t - s_k) + \sum_{k=1}^{n-1} P_k m(t - s_k), \qquad (0 < t \le s_n) \tag{3.13}$$

and can be solved by the same way as the equation (3.10).

3.4 Numerical examples

In this section, we see some numerical example to research how the reliability degrades under the history dependent inspection policy. For this purpose, it is necessary to fix the forms of the probability density function of initial crack length, $g(x)$, and the probability of detection (POD) of an existing crack, $D(x)$.

Firstly, we make an assumption that the initial crack length obeys the Weibull distribution, as shown in the fig. 3.2.

Secondary, $D(x)$ is fixed to the following model. Koul *et al.* researched the several types of non-destructive inspection on the compressor disk of the aircraft engine, and adjusted their POD curve into the following form:

$$D(x) = \frac{\exp(\beta_0 + \beta_1 \ln x)}{1 - \exp(\beta_0 + \beta_1 \ln x)}, \tag{3.14}$$

where β_0 and β_1 are constants. Figure 3.3 shows an example of POD curve by the penetrating inspection method.

The Subsequent investigation on the reliability degradation is argued by means of the probability distribution function of the initial crack length as shown in the fig. 3.2 and the POD curve in the fig. 3.3. Here, the number of loading cycles are replaced by the variable t defined in the eqn. (3.3). The variable t is proportional to the number of loading cycles, but it is noted that it corresponds to the less number if the mean propagation resistance $E[C_n]$ or the stress amplitude Z_0 becomes to larger value. Figures 3.4 and 3.5 shows the time variation of the failure probability $H(t) = 1 - R(t)$ by the log scale, and that of the mean inspection times, respectively. The symbols A, B and C in the figures corresponds to the inspection plans in the Table 3.1.

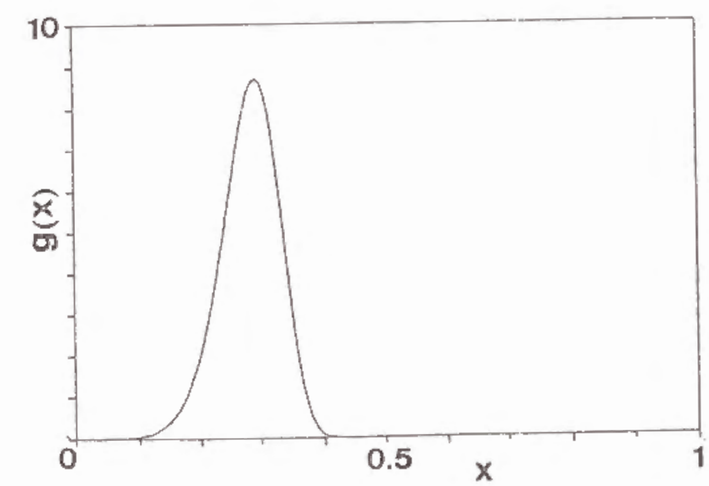

Fig. 3.2. The probability density function of the initial crack length.

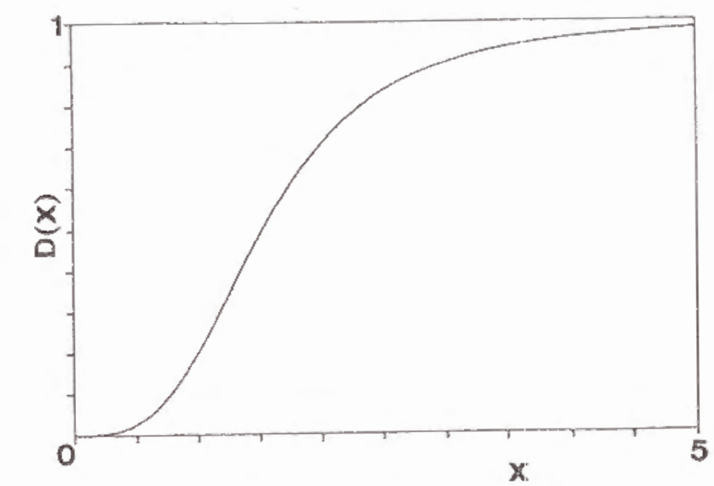

Fig. 3.3. The crack detection probability as a function of crack length.

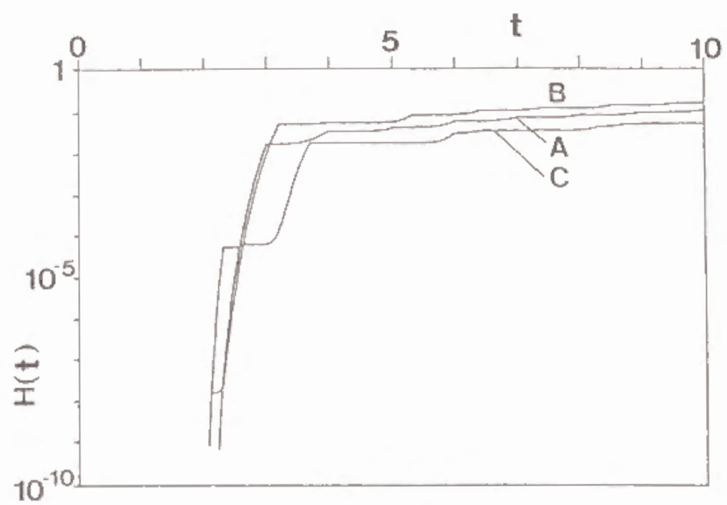

Fig. 3.4. The failure probability curves.

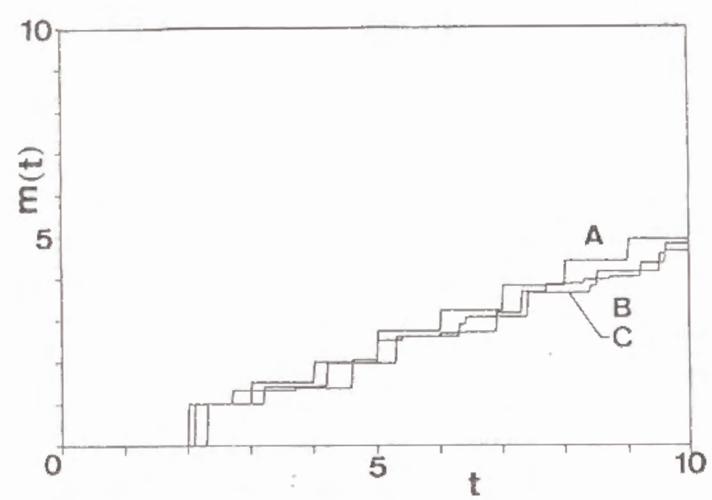

Fig. 3.5. The mean number of inspections.

plan	t_1	t_2	t_3	t_4
A	2.0	1.0	1.0	1.0
B	2.1	1.1	0.8	0.4
C	2.3	0.4	1.0	0.5

Table 3.1. The inspection plans.

The inspection intervals should be prespecified so that the sum of them are equal to or greater than the total inspection periods. In this example, however, the intervals $t_5, t_6, \cdots$ are omitted from numerical calculation for reason of the negligibility of the value of $f(t)$ such that $t > t_4$ in comparison to that of $R(t)$. This means the fact that it has almost no possibility that the events after t_4 occurs, that is, there is almost no components which pass in the all inspections of past four times, and they either failed or exchanged. The parameters of crack propagation are settled as $\lambda = 0.5$, $\alpha_0 = 0.32$ and $\zeta_c = 0.1$, respectively, and the critical crack length, $x_c = 40$.

Figures 3.4 and 3.5 tell us the plan C is most effective plan in the three in both points of the reliability and the mean inspection times. As shown in this example, in the maintenance of the structural component along this model in which the component is exchanged only if any cracks are detected, in order to raise the efficiency of the inspection, it is the most important point to decide the inspection interval such that the crack propagates up to adequate length to be able to detect and not to overlook the large cracks which cause the failure of the components. So, the inspection intervals are decided by the balance of both the ability of crack detection and the distribution of crack length. The advantage of the plan C has originated from the shorter interval t_2 than others, so that the large cracks which are overlooked at the first inspection are detected at the second inspection before failure occurs.

3.5 Conclusions

The discussions in this chapter are argued on the development of the estimation method for the structural reliability and mean inspection times for the history-dependent inspection policy, on the basis of the stochastic crack propagation model in consideration of the randomness of the crack propagation resistance. As the result, both the reliability and the mean inspection times are calculated with an idea of the reproductive process because of the recurrence of the inspection intervals. The histry-dependent inspection policy to be considered in this chapter has the advantage of the fixed interval inspection in the point of less number of inspection times. However, if the histry-dependent inspection policy is applied to the many components in the structure, the inspections may come in succession, so the service stops many times for increasing inspections and causes the sirious degradation of the availability of the structure. It seems to be effective in case that the inspection is performed, chiefly, on the special important component in the structure, or on the component which is considered to fail at first with the prediction.

For the history-dependent inspection policy, it is difficult to find the most efficient plan because the inspection times becomes a kind of stochastic process and evaluated as a expected value.

References

[1] H. Kitagawa and T. Hisata, *Trans. of JSME* (in Japanese), **A-45** (1979), 1033.

[2] Y. Shimada, T. Nakagawa and H. Tokuno, *J. of JSMS* (in Japanese), **34** (1985), 327.

[3] H.O. Madsen, J.D. Sorensen and R. Olesen, *Proc. ICOSSAR'89*, **3** (1990), 2099.

[4] Y. Fujimoto, H. Itagaki, S. Itoh, H. Asada and M. Shinozuka, *Proc. ICOSSAR'89*, **3** (1990), 2143.

[5] Y. Fujimoto, A.M. Swilem and M. Iwata, *Proc. SNAJ*, **168** (1990), 489.

[6] A. Tsurui and A. Sako, *CJMR* Vol.**5** (1989) p.153, Elsevier.

[7] A. Tsurui, A. Sako and T. Tanaka, *J. of JSMS* (in Japanese), **39** (1990), 748.

[8] T. Tanaka and A. Tsurui, *Structural Safety*, **9** (1991), 305.

[9] A. Tsurui and H. Ishikawa, *Trans. of JSMS* (in Japanese), **A-51** (1985), 461.

[10] A. Tsurui and H. Ishikawa, *Structural Safety*, **4** (1986), 15.

[11] A.K. Koul. N.C. Bellinger and A. Fahr, *Int. J. Fatigue*, **12** (1990), 379.

Chapter IV

A Probabilistic Approach to the Random Propagation of Two Collinear Cracks

4.1 Introduction

In the structural component which has plural cracks, it sometimes happens that the simultaneously growing cracks join together to cause the fatigue failure of the component. In the situation that cracks join, in addition to the degradation of the static strength, the interaction between cracks accelerates their growth speed more rapidly as they are closer. So the influence of the interaction between cracks are not negligible if large cracks are sited in near position. Of course, the critical crack length depends on the mutual distance between the cracks, which is much less than that of the single crack. The type of this damage is called Multi Sited Damage (MSD), which sometimes becomes a cause of the fracture of aircraft [1].

Generally, it is well-known that the fatigue crack has a random property in the propagation even if the environment is identical. Therefore the probability theory is required to describe the fatigue crack growth and to estimate the life up to joining. Okada *et al.* [2]. reported the life distribution of the metallic specimen with two cracks through many experiments, and they also examined by the model in probability theory. However, in their model lacks the randomness in the crack growth process. Even if we have the information of the initial flaw in the material, the speed of the crack growth is considerably influenced by the uncertainty from material inhomogeneity, and show the random property To describe the random crack growth, we make use of the stochastic crack growth model.

Tsurui *et al.* derive the theoretical distribution of the crack length from the Markov approximation method for random crack growth, and evaluate the life distribution for a single fatigue crack [3-6]. In order to analyze the life of MSD, we take account of the interaction of the cracks. Therefore it is necessary to describe MSD as multi variate stochastic process. In this chapter, the stochastic behavior of two collinear cracks is investigated by means of the Markov approximation method applied to two variate stochastic process.

According to the application of Markov approximation method, the crack length distribution function is expressed as a solution of the time variant partial differential equation with two spatial variables. In order to solve the equation, we can choose two methods, one the analytic approximation, and the other, the numerical analysis. If we choose the analytic approximation, there is a benefit in the evaluation of the reliability, but the accuracy is lost as the time proceeds. Conversely, the numerical calculation through the difference scheme for the partial differential equation cannot be always expected the accuracy, while the general behavior of the solution is easily observed. In the following sections, in order to combine the advantage of both methods, the analytic approximation is applied in the earlier stage of the crack growth, and the numerical method in the later stage, for the purpose of the investigation of the crack length distribution and the life distribution up to crack joining.

4.2 The crack growth model

In this section, the two variate stochastic crack growth model, which is utilized in the later sections, is introduced through the Markov approximation by Tsurui *et al.* The object of the analysis is the propagation of the two collinear cracks, as is shown in fig. 4.1, under the constant amplitude. Figure 4.1 describes the two cracks which have different length a and b with distance d between their center holes. Our main concern is the time up to their joining. Each stress intensity factor at the four crack tips is different, according to the crack length, so the growth speeds are not identical in general. Here, we concerns the stress intensity factor at the two points in the inside of both cracks, and make an assumption that the points are symmetry to inner points in the approximation. This assumption is valid for the earlier stage of the crack growth if the length of the

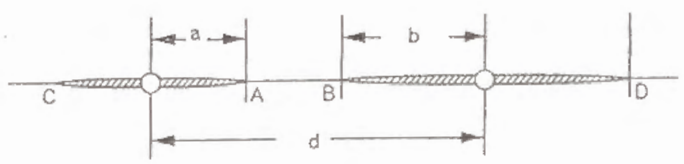

Fig. 4.1. Collinear cracks.

cracks are small, for instance, the growth ratio of the outer crack tip versus inner crack tip is 0.7 if $a/d = b/d = 0.25$, and 0.95 if $a/d = b/d = 0.4$, under the condition that the exponent of Paris' law is equal to 3. If the approximation is not allowed, the reliability estimation is not in danger side at least. From this reason, we make assumptions that the distance d does not change and that the progress of the outer crack tips are neglected.

If the Paris's law is applicable to both crack growth with identical exponent m, the each crack growth obeys,

$$\frac{da}{dn} = \varepsilon C_A(n)\left(\frac{\Delta K_A}{K_0}\right)^m, \tag{4.1}$$

and

$$\frac{db}{dn} = \varepsilon C_B(n)\left(\frac{\Delta K_B}{K_0}\right)^m, \tag{4.2}$$

where ΔK_A and ΔK_B are the ranges of the stress intensity factor at the points A and B, respectively, and K_0 stands for the constant stress intensity factor. In above equations, we make an assumption that C_A and C_B are statistically independent and locally stationary stochastic process which obey the identical probability distribution.

The stress intensity factors for the at collinear cracks in infinite body which is stretched by the uniform load σ are expressed as follows [7]:

$$K_A = \sigma\sqrt{\pi}\sqrt{\frac{d-a+b}{a(d-a-b)}}\left\{a - \frac{1}{2}(d-b+a)\psi(\lambda)\right\}, \tag{4.3}$$

$$K_B = \sigma\sqrt{\pi}\sqrt{\frac{d-b+a}{b(d-b-a)}}\left\{b - \frac{1}{2}(d-a+b)\psi(\lambda)\right\}, \tag{4.4}$$

where

$$\lambda = 2\sqrt{\frac{ab}{(d-a+b)(d+a-b)}}, \tag{4.5}$$

$$\psi(\lambda) = 1 - \frac{E(\lambda)}{K(\lambda)}, \tag{4.6}$$

$$E(\lambda) = \int_0^{\pi/2}\sqrt{1-\lambda^2\sin^2\theta}\,d\theta, \tag{4.7}$$

$$K(\lambda) = \int_0^{\pi/2}\frac{d\theta}{\sqrt{1-\lambda^2\sin^2\theta}}. \tag{4.8}$$

$E(\lambda)$ and $K(\lambda)$ mean the first and the second kind of the elliptic integral, respectively. For the numerical calculation, if $\lambda < 0.9$, the function $\psi(\lambda)$ is enough precisely approximated by the Maclaurin expansion

$$\psi(\lambda) = \frac{1}{2}\lambda^2 + \frac{1}{16}\lambda^4 + \frac{1}{32}\lambda^6 + \frac{41}{2048}\lambda^8 + \cdots, \tag{4.9}$$

which is truncated by the fourth term.

For the normalization, let $X = a/d$ and $Y = b/d$, and rewrite ε/d with ε.

Substituting the eqns (4.3) and (4.4) into (4.1) and (4.2), we get the following expressions:

$$\frac{dX}{dn} = \varepsilon C_A f(X,Y) \tag{4.10}$$

$$\frac{dY}{dn} = \varepsilon C_B g(X,Y) \tag{4.11}$$

$$f(X,Y) = \left[\sqrt{\frac{1-x+y}{x(1-x-y)}}\left\{x - \frac{1}{2}(1-y+x)\psi(\lambda)\right\}\right]^m \tag{4.12}$$

$$g(X,Y) = \left[\sqrt{\frac{1-y+x}{y(1-y-x)}}\left\{y - \frac{1}{2}(1-x+y)\psi(\lambda)\right\}\right]^m \tag{4.13}$$

$$\lambda = 2\sqrt{\frac{xy}{(1-x+y)(1+x-y)}}, \tag{4.14}$$

Obviously, the symmetric relation $f(x,y) = g(y,x)$ holds.

When the Markov approximation method [3,4] is applied to the above equations, $w(x,y,n)$, that is, the joint probability density function of (X,Y) at the n-th cycle of loading, is described as the solution of the following Fokker-Planck equation:

$$\frac{\partial w}{\partial t} = -\frac{\partial}{\partial x}fw - \frac{\partial}{\partial y}gw + s_x\frac{\partial}{\partial x}\left\{f\frac{\partial}{\partial x}fw\right\} + s_y\frac{\partial}{\partial y}\left\{g\frac{\partial}{\partial y}gw\right\}, \tag{4.15}$$

where,

$$t = \mu_c n, \tag{4.16}$$

$$s_x = \frac{\sigma_c^2}{\mu_c^2}\tau_c(t,x_0), \tag{4.17}$$

$$s_y = \frac{\sigma_c^2}{\mu_c^2}\tau_c(t,y_0). \tag{4.18}$$

The parameters μ_c and σ_c are the expectation and the variance for the εC_A (and εC_B), respectively. The function τ_C corresponds to the correlation cycle of the crack growth resistance converted from the correlation distance α_0, which is approximated by

$$\tau_C(t,x_0) = \begin{cases} \alpha_0\left\{x_0^{-(m/2-1)} - \left(\dfrac{m}{2}-1\right)t\right\}^{\frac{m}{m-2}}, & \left(t < \dfrac{x_0^{-(m/2-1)}}{m/2-1}\right), \\ 0, & \left(t \geq \dfrac{x_0^{-(m/2-1)}}{m/2-1}\right), \end{cases} \tag{4.19}$$

It is noted that the coefficients s_x and s_y depend on the initial crack length x_0 and y_0.

4.3 Approximate solution of the Fokker-Planck equation

In general, it is not always possible to find the analytical solution of the two dimensional Fokker-Planck equation in the preceding section. Therefore, we have to apply the approximation to the equation, or to calculate the solution numerically, in order to evaluate $w(x,y,n)$. The numerical calculation, however, comes to the difficulty at the initial state, since $w(x,y,n)$ at the initial state is the Dirac's delta function if the initial state is deterministic. For this reason, the it is necessary to constitute an approximate solution in the analytic manner in the early stage of the process, even if we make use of the numerical solution of difference scheme. In this section, An analytic approximate solution is proposed, by means of the trajectory approximation on the eqns (4.10) and (4.11).

Prior to enter the main discussion, we observe the behavior of the stress intensity factor in the eqns (4.3) and (4.4). Figure 2 shows the contour of K_A, that is, the stress intensity factor at the point A. From this figure, the contours are nearly parallel to the Y axis in the region $X+Y<0.5$. This means that the stress intensity factor is not influenced by the far crack, if the distance is enough greater than the length of both cracks. So, if X and Y are interpreted as stochastic processes, they are independent in the region $X+Y<0.5$.

On the other hand, when $X+Y$ is close to unity, K_A and K_B rapidly increase, and, when $X+Y=1$, both diverge. However, in the physical reality, if the tips of both cracks are too close, the both join together and cause the failure instantaneously. Therefore the eqns (4.3) and (4.4) are valid only in a certain region. So we make the assumption that the eqns (4.3) and (4.4) is applicable in $X+Y<L$ $(L=0.8\sim0.9)$, and that, if the sum exceeds L, the cracks join together. The probability density are calculated is limited in $X+Y<L$. This assumption does not based on the mechanical behavior of the fracture, but is useful in the practical purpose, for the assumption leads the life estimation to safety side. The growth model is described simply in the linear fracture mechanics under this assumption.

The central trajectory of (X,Y) satisfies the following relation:

$$\frac{dx}{dy}=\frac{E[C_A]f(x,y)}{E[C_B]g(x,y)}. \tag{4.20}$$

Giving an adequate initial value (x_0,y_0), we get the trajectory by the solution of the above equation, that is,

$$x=q_1(y), \tag{4.21}$$

or

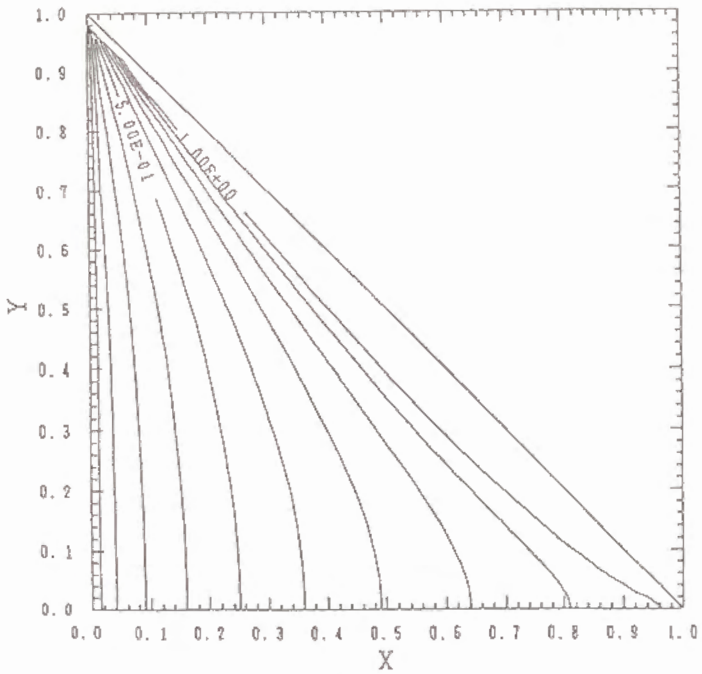

Fig. 4.2. Contour of K_A.

$$y = q_2(x). \tag{4.22}$$

They express the same curve. The stochastic process (X, Y) fluctuates randomly around the central trajectory curve. Here we make an approximation by replacing $f(X, Y)$ with $f(X, q_2(X))$ and $g(X, Y)$ with $g(Y, q_1(Y))$ in the eqns (4.10) and (4.11). By this approximation, the increment of X and Y become independent, owing to the independence between C_A and C_B. This approximation is not so bad in the early stage of the process, for X and Y are independent if they are small, as previously mentioned.

On the basis of this idea, we apply the Markov approximation method to the approximate growth equation,

$$\frac{dX}{dn} = \varepsilon C_A f(X, q_2(X)), \tag{4.23}$$

$$\frac{dY}{dn} = \varepsilon C_B g(q_1(Y), Y). \tag{4.24}$$

According to Ref.[5], the Fokker-Planck equation associated with the above equations can be solved and expressed as

$$w(x,y,t) = \frac{1}{4\pi \tilde{f}(x)\tilde{g}(y)\sqrt{G_A(t,x_0)G_B(t,y_0)}} \times \exp\left\{-\frac{\left[\int_{x_0}^{x}\frac{dx}{\tilde{f}(x)} - t\right]}{2G_A(t,x_0)}\right\} \exp\left\{-\frac{\left[\int_{y_0}^{y}\frac{dy}{\tilde{g}(y)} - t\right]}{2G_B(t,y_0)}\right\} \tag{4.25}$$

where,

$$\tilde{f}(x) = f(x, q_2(x)), \tag{4.26}$$

$$\tilde{g}(x) = g(q_1(y), y), \tag{4.27}$$

$$G_A(t,x_0) = \begin{cases} \dfrac{\alpha_0}{m-1}\left[x_0^{-(m/2-1)} - \left\{x_0^{-(m/2-1)} - \left(\dfrac{m}{2}-1\right)t\right\}^{\frac{2m-2}{m-2}}\right], & \left(t < \dfrac{x_0^{-(m/2-1)}}{m/2-1}\right), \\ \dfrac{\alpha_0}{m-1} x_0^{-(m/2-1)}, & \left(t \ge \dfrac{x_0^{-(m/2-1)}}{m/2-1}\right), \end{cases} \tag{4.28}$$

$$G_B(t,y_0) = \begin{cases} \dfrac{\alpha_0}{m-1}\left[y_0^{-(m/2-1)} - \left\{y_0^{-(m/2-1)} - \left(\dfrac{m}{2}-1\right)t\right\}^{\frac{2m-2}{m-2}}\right], & \left(t < \dfrac{y_0^{-(m/2-1)}}{m/2-1}\right), \\ \dfrac{\alpha_0}{m-1} y_0^{-(m/2-1)}, & \left(t \ge \dfrac{y_0^{-(m/2-1)}}{m/2-1}\right), \end{cases} \tag{4.29}$$

The failure probability at the time t are calculated from

$$P_F(t) = 1 - \int_{D_{SAFE}} w(x,y,t)\,dxdy, \tag{4.30}$$

where D_{SAFE} means the non-failure region,

$$D_{SAFE} = \{(x,y) | x, y > 0, x + y < L\}. \tag{4.31}$$

By the way, another problem arises in the decision of the central trajectory from the shape of the region. Though the right hand side of the eqns. (4.23) and (4.24) must be defined in the all point in D_{SAFE}, the central trajectory does not exist in $X + Y > L$. Suppose that the (x_c, y_c) be the intersection point of the trajectory and $X + Y = L$, which depends on the initial value (x_0, y_0), the eqns. (4.23) and (4.24) are not determined if $X > x_c$ or $Y > y_c$. Thence, we extend the trajectory to outside the region as shown in fig.4.3. On the revision of the trajectory, the following two points are required.

(i) The mean difference between revised trajectory and the true solution process must be small.

(ii) The approximated growth velocity is not estimated as smaller value than that of the mean value of the true solution process.

Taking notice of the relation

$$f(X,Y) \le f(X, L - X), \tag{4.32}$$

$$g(X,Y) \le g(L - Y, Y), \tag{4.33}$$

holds in D_{SAFE}, we constitute the virtual trajectory so that the value on the trajectory is equal to the maximum value. For the purpose, we define q_1 and q_2 in the outer region as

$$q_1(y) = L - y, \quad (y > y_c), \tag{4.34}$$

$$q_1(x) = L - x, \quad (x > x_c), \tag{4.35}$$

Conversely, in the region $X < x_0$ or $Y < y_0$, there exist the lower limit point, where the trajectory intercepts the X axis or Y axis and the value of the eqn. (4.21) or (4.22) is not determined. We avoid this problem by interpreting the probability as zero in such region. Of course, the decrease of the X or Y cannot occur in the actual growth process. The result of Markov approximation produces a slight probability of decrease, but the error is negligible for practical use, since the probability of decrease becomes smaller and smaller as the time proceeds.

4.4 Difference scheme for Fokker-Planck equation

The equation (4.15) has the same form as the drift diffusion equation, so we can make use of the several numerical calculation techniques in fluid mechanics. But variable coefficients in the eqn. (4.15) bring about some problems in the numerical calculation. (i) The coefficients vary from zero to infinity with respect to the spatial variable, so that, the stable region is restricted in accordance with numerical precision. (ii) The coefficients of diffusion terms decrease as the time proceeds. So the computational error

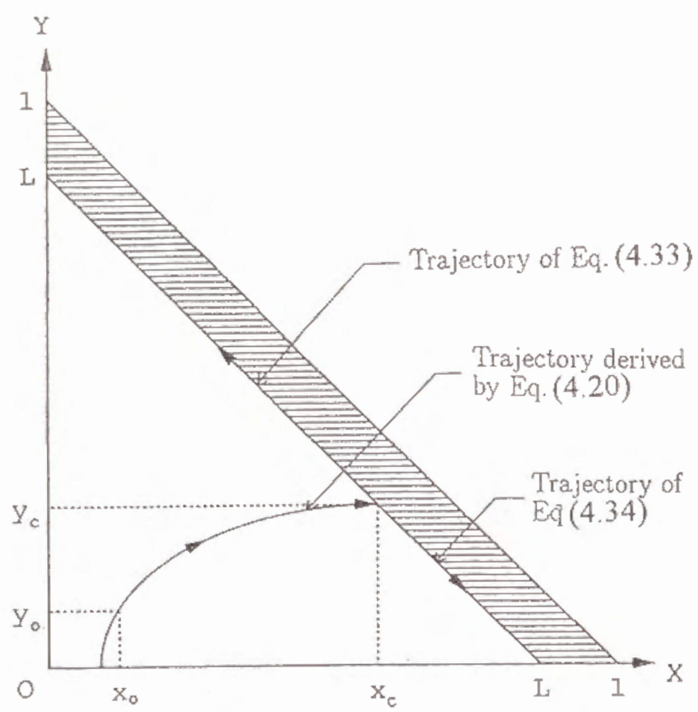

Fig. 4.3. Revision of the mean trajectory of $(X(t), Y(t))$.

increases, since the drift terms become relatively strong to the diffusion terms. For these reasons, the numerical precision and the computational stability are not always compatible. To avoid these problems, the Euler's method is adopted so as to determine the stable region, and the approximate solution in the preceding section is used in the early stage.

In this study, the difference scheme is applied as follows. Let $x_i = ih$, $y_j = jh$ and $t_k = k\tau$, $(i, j, k = 1, 2, \cdots)$, where h is the width of division for x and y, and τ for the t. The equation (4.12) is converted to

$$
\begin{aligned}
w_{i,j}^{k+1} = w_{i,j}^{k} &- \frac{\tau}{h}(f_{i,j}w_{i,j}^{k} - f_{i-1,j}w_{i-1,j}^{k}) - \frac{\tau}{h}(g_{i,j}w_{i,j}^{k} - g_{i,j-1}w_{i,j-1}^{k}) \\
&- \frac{s_x\tau}{h^2}\left\{f_{i+\frac{1}{2},j}f_{i+1,j}w_{i+1,j}^{k} - \left(f_{i+\frac{1}{2},j}f_{i,j} + f_{i-\frac{1}{2},j}f_{i,j}\right)w_{i,j}^{k} + f_{i-\frac{1}{2},j}f_{i-1,j}w_{i-1,j}^{k}\right\} \\
&- \frac{s_y\tau}{h^2}\left\{g_{i,j+\frac{1}{2}}g_{i,j+1}w_{i,j+1}^{k} - \left(g_{i,j+\frac{1}{2}}g_{i,j} + g_{i,j-\frac{1}{2}}g_{i,j}\right)w_{i,j}^{k} + g_{i,j-\frac{1}{2}}g_{i,j-1}w_{i,j-1}^{k}\right\}
\end{aligned}
\tag{4.36}
$$

where,

$$w_{i,j}^{k} = w(x_i, y_j, t_k) \tag{4.37}$$

$$f_{i,j} = f(x_i, y_j) \tag{4.38}$$

$$g_{i,j} = g(x_i, y_j) \tag{4.39}$$

For the simplicity, L is set to a multiple of the h. The range of the subscripts are defined by

$$I = \{(i,j) | i, j > 0, i + j < L/h\}. \tag{4.40}$$

To keep the stability of the difference scheme, the parameters should be selected so as to satisfy the following condition in the domain $x_i + y_j < L$,

$$
\begin{aligned}
&\frac{\tau}{h}(f_{i,j} + g_{i,j}) + \frac{s_x\tau}{h^2}\left(f_{i+\frac{1}{2},j}f_{i,j} + f_{i-\frac{1}{2},j}f_{i,j}\right) \\
&+ \frac{s_y\tau}{h^2}\left(g_{i,j+\frac{1}{2}}g_{i,j} + g_{i,j-\frac{1}{2}}g_{i,j}\right) < 1
\end{aligned}
\tag{4.41}
$$

The above condition is evaluated with the maximum value for s_x and s_y with respect to the time t.

The boundary condition is imposed by the following reflective condition on $x = 0$ and $y = 0$,

$$\frac{\tau}{h}f_{i,j}w_{i,j}^{k} - \frac{s_x\tau}{h^2}\left(f_{i+\frac{1}{2},j}f_{i+1,j}w_{i+1,j}^{k} + f_{i+\frac{1}{2},j}f_{i,j}w_{i,j}^{k}\right) = 0, \quad (i = 0), \tag{4.42}$$

$$\frac{\tau}{h}g_{i,j}w_{i,j}^{k} - \frac{s_y\tau}{h^2}\left(g_{i,j+\frac{1}{2}}g_{i,j+1}w_{i+1,j}^{k} + g_{i,j+\frac{1}{2}}g_{i,j}w_{i,j}^{k}\right) = 0, \quad (j = 0), \tag{4.43}$$

and the absorbing condition on $x + y = L$,

$$w_{i,j} = 0, \quad (i + j = L / h). \tag{4.44}$$

The total probability decreases, owing to eqn. (4.44).

The initial condition is given by the following condition at a certain time $t = t_0$.

$$w_{i,j}^{k_0} = w(x, y, t_0) \tag{4.45}$$

where $w(x,y,t_0)$ is an approximate solution derived by eqn. (4.25). The initial time t_0 should be determined with attention to the two point. (i) At $t = t_0$, the probability for $X + Y > 0.5$ is sufficiently small. (ii) The peak of the initial distribution is not very sharp.

The failure probability is calculated through the trapezoid formula, that is,

$$P_F = 1 - \sum_{(i,j)\in I} w_{i,j}\,. \tag{4.46}$$

4.5 Result of the numerical calculation

From the approximate solution in section 4.3 and the difference scheme in section 4.4, the numerical calculation is tried as an example in this section. Both are conjoined at the time $t_0 = 3.0$ when the initial density is almost implied in the domain $X + Y < 0.5$. The quantization parameters are $h = 0.01$ and $\tau = 0.0025$, the material parameters $\sigma_C / \mu_C = 0.3$, $\alpha_0 = 0.01$, $m = 3.0$, and the limit state parameter $L = 0.8$. Under this situation, the joint density function for crack density and failure probability are evaluated for three cases of initial crack length, (x_0, y_0) = (0.05, 0.05), (0.04, 0.06), and (0.03, 0.07).

Figures 4.4 and 4.5 show the contours of the crack length density function at $t = 4.5$ for the initial length (0.05, 0.05) and (0.04, 0.06), respectively. Each contour is expressed with log scale, that is, $w = 0.05 \times 2^i$, $(i = 0, 1, 2 \cdots, 9)$. Figure 4.6 shows the log scaled failure probability as a function of the time for the initial length (0.05, 0.05), (0.04, 0.06), and (0.03, 0.07), respectively.

From figs. 4.4 and 4.5, it is observed that the difference of the initial crack length has a great influence on the shape of the density function. The tendency of growth of both cracks is unstable, that is, the small difference of their length at the initial stage is magnified as the time proceeds. Figure 4.6 also tells us that the greater difference of the initial crack length brings about the larger failure probability, even if the sum of the both crack lengths are kept unchanged. In the other words, the variance of the life distribution becomes larger value if the initial difference is greater. These are acceptable results in the view point of the deterministic fracture mechanics. Therefore, it is concluded that the method used in this chapter is a good approximation to the probabilistic approach in the problem of two collinear cracks.

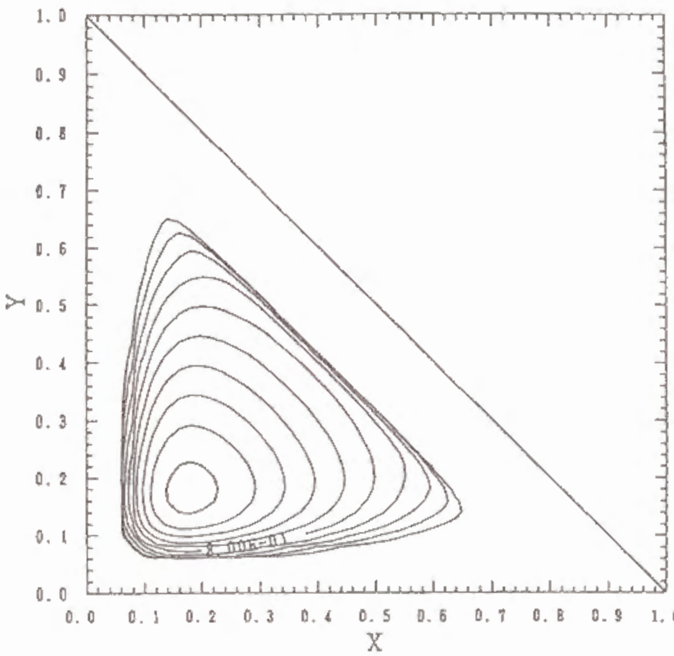

Fig. 4.4. Contour of the w when (x_0, y_0) =(0.05, 0.05).

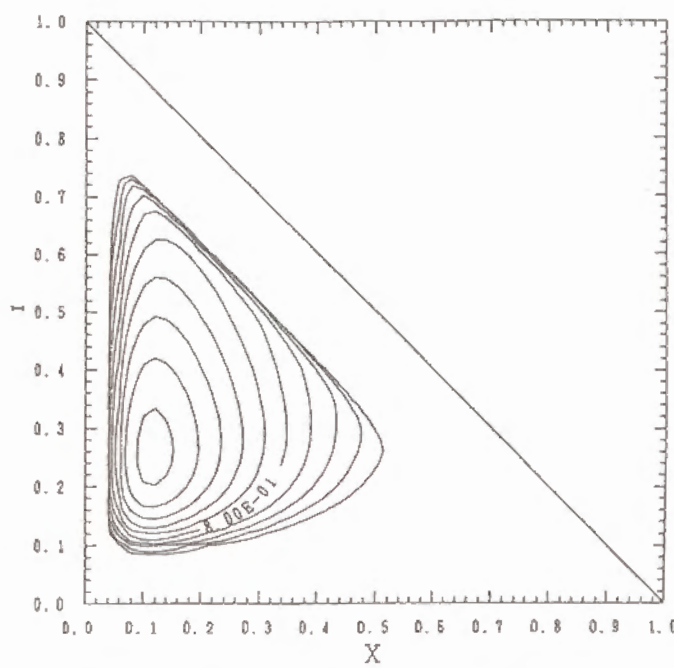

Fig. 4.5. Contour of the w when (x_0, y_0) =(0.04, 0.06).

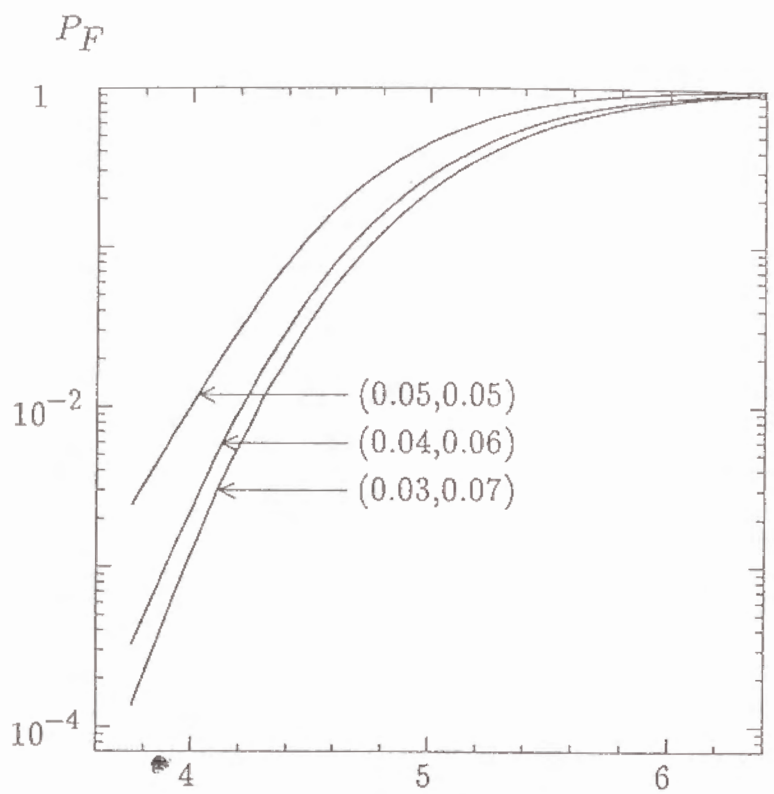

Fig. 4.6. The failure probability.

4.6 Conclusion

In this chapter, the Markov approximation method was applied to the problem of the random growth of two collinear cracks. The growth process was modeled by the two variate stochastic process which includes the interaction between the cracks. On the evaluation of the probability density function and the failure probability were evaluated through the combination of the both analytical and numerical approximation to solve two-dimensional Fokker-Planck equation. Lastly, the sensitivity with respect to the initial crack length was examined by the numerical example. It is observed that the difference of the initial crack length magnifies the failure probability.

References

[1] C.E. Feddersen, Evaluation and Prediction of the Residual Strength of Center Cracked Tension Panel, ASTM STP **486** (1970).

[2] K. Okada and T. Tanaka, *J. of JSMS* (in Japanese), **38** (1980), p.130.

[3] A. Tsurui and H. Ishikawa, *Trans. of JSME* (in Japanese), **A-51** (1985), p.461.

[4] A. Tsurui and H. Ishikawa, *Structural Safety*, **4** (1986), 15-29.

[5] H. Tanaka, *Engrg. Frac. Mech.*, **34** (1989), p.189.

[6] H. Tanaka, A. Tsurui, S. Takagi, *Proc. of JCOSSAR'91* (in Japanese), p.587.

[7] M. Ishida, The elastic analysis of crack and stress intensity factor, (in Japanese), 1989, Baifu-kan, Tokyo.

[8] T. Murata, R. Natori, Y. Karaki, Large scale numerical simulation (in Japanese), 1990, Iwanami-Shoten, Tokyo.

Chapter V

An Analytical Method for Leak Before Break Assessment Based upon Stochastic Fracture Mechanics

5.1. Introduction

Leak Before Break (LBB) assessment in piping systems or pressure vessels is one of the most interesting themes in structural safety, especially in the safety assessment of nuclear power plants. To ensure a highly reliable safety, we must take into consideration the fact that there are many uncertain factors with respect to the structural component's resistance and service environment [1]. That is to say, the LBB as well as the structural safety should be assessed from a view point of the structural reliability.

In order to achieve a probabilistic LBB assessment for piping system from a view point of the structural reliability, many studies have been reported until the present stage, in which the following uncertain or random factors have been included in the analysis [2]. uncertainty on the number of initial flaws [3], uncertainty on the initial crack state (initial crack depth and initial aspect ratio) [3,4], probability of crack-detection and leak-detection [5,6] and uncertainty on the physical parameters of materials such as fracture toughness. However, few studies have treated the effect of random loading history and/or spatially random configuration of the material's resistance against the surface crack propagation. Whether the LBB is realized or not depends on the uncertainty factors investigated until the present stage, but also on the random nature with respect to crack propagation, such as randomness o material's resistance or random loading history, since they undoubtedly have an affect on the crack growth process. To take into analysis the later ones, we must treat the crack growth process not as a simple random variable but as a stochastic process. That is, we must derive the probability distribution on the crack size based upon the fact that the crack growth process shows a temporally random property according to the random variation of the material's resistance and/or loading. Thus, we need to obtain a probability distribution for the surface crack propagation by taking such uncertainties simultaneously into account.

H. Tanaka *et al.* have already reported some studies concerning the stochastic propagation of surface cracks with the aid of the so-called stochastic fracture mechanics, which mathematically treats the crack growth equation as a stochastic differential equation reflecting the temporally random variation of the material's property and/or loading. In these studies, we have utilized an extended Markov approximation method [7] to obtain a probability distribution of the surface crack propagation. In references [8] and [9], by the use of an approximate propagating equation such that the well-known through crack propagating equation is directly applied to each direction, we have succeeded to obtain a probability density function and the loading amplitude are random. However, since we used such an approximate propagating equation, the variation of the aspect ratio has not been well reproduced for some values of parameters. In order to improve this point, in reference [10], we revised the propagating equation by evaluating the stress intensity factor of the surface crack with the aid of the Newman-Raju's solution [11], assuming two points: that (a) the specimen's size is sufficiently large compared with the crack size and that (b) the material's property is not random, we also succeeded in obtaining a probability density function in an analytical form by the use of this improved propagating equation.

In this chapter, extending the stochastic model used in reference [10], we investigate the stochastic growth process of surface cracks and discuss a way to apply it to the probabilistic LBB assessment for piping system. Basically, we follow the assumptions used in the previous studies [8-10] such that: (i) the surface crack is always semi-elliptical until it penetrates the specimen's thickness or width, (ii) its propagation can be decomposed into surface and depth directions, (iii) the well-known Paris-Erdogan's law [12] is applicable to each direction.

The main purposes of this study are the following two points: one is to obtain a

probability distribution of the surface crack propagation in an analytical form without making the two assumptions used in reference [10], the other is to propose an analytical method which utilizes the obtained probability distribution of the surface crack growth to the LBB assessment for piping systems. First, we mathematically formulate the stochastic surface crack propagation and derive its probability distribution by utilizing a Markov approximation method [7], under the condition that (i) both the loading process and materials' property are random, and (ii) the specimens size is finite. Next, we mathematically define the probability of LBB in piping systems, and propose the way of its derivation by the use of the probability distribution of the surface crack propagation.

5.2. Growth equation of surface cracks

5.2.1 Basic equation

In this chapter, we consider a semi-elliptical surface crack in a finite plate growing under uniform tensile stress, which is illustrated in fig.5.1, in which $A_1(n)$, $A_2(n)$, B_1, B_2 represents a half surface length after n cycles of loading, a crack depth after n cycles, a specimen's half width and a specimen's thickness, respectively. This can be a simple model of a surface crack growing from the internal surface of a pipe, if its internal radius is sufficiently large compared with its thickness. That is to say, we can consider that the plate is approximately equivalent to a pipe with internal radius B_1/π and thickness B_2 with respect to the surface crack growth (this is schematically illustrated in fig.5.2).

As mentioned in the introduction, we assume that the surface crack is always semi-elliptical until it penetrates the specimen's thickness or width, which means that the crack propagation can be mathematically described as a bivariate process. As a propagating law of the surface crack, we use the well-known is of Paris-Erdogan's one [11,12], i.e.,

$$\frac{dA_j}{dn} = \varepsilon_{0j}\left(\Delta K_j\right)^{2(\lambda+1)} \quad (j=1,2), \tag{5.1}$$

where ΔK_j represents a stress intensity factor range at the point P_j in fig.5.1, and ε_{01}, ε_{02}, λ are respectively material constants (in which ε_{01}, ε_{02} will be transformed into random variables in later discussion, see Section 5.3). It should be noted that λ usually takes on a positive value for ordinary metallic materials.

Applying the well-known Newman-Raju's solution [11], which has been

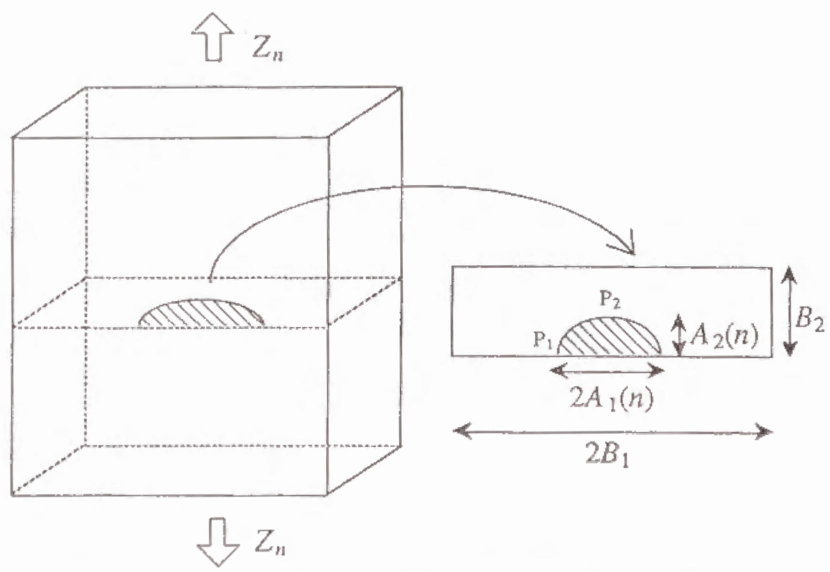

Fig. 5.1. Semi-elliptical surface crack in a plate.

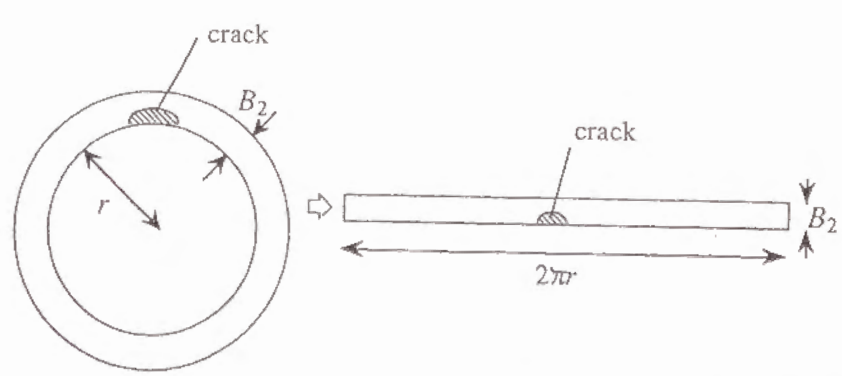

Fig. 5.2. Conceptional illustration on a modeling of a surface crack in a pipe.

obtained through a three dimensional stress analysis by the use of a finite element method, to the stress intensity factor of the surface crack, and transforming the dependent variables (A_1, A_2) into non-dimensional ones by $X_j = A_j / B_j \quad (j = 1,2)$, we can obtain the following simultaneous differential equations describing crack growth:

$$\frac{dX_j}{dn} = \varepsilon_j Z_n^{2(\lambda+1)} g_j(X_1, X_2) \quad (j = 1,2), \tag{5.2}$$

in which Z_n represents an adequately normalized tensile stress amplitude at n-th cycle of loading, ε_j are defined as $\varepsilon_j = \varepsilon_{0j}\pi^{\lambda+1}B_2^{\lambda} \quad (j = 1,2)$, and the functions g_j (j = 1,2) are defined as follows:

$$g_1(X_1, X_2) = B^{*\lambda+2}\left[\frac{M(X_1, X_2)}{\sqrt{Q(X_1, X_2)}}\frac{X_2}{\sqrt{X_1}}g(X_2)f_w(X_1, X_2)\right]^{2(\lambda+1)}, \tag{5.3.a}$$

$$g_2(X_1, X_2) = B^{*\lambda+2}\left[\frac{M(X_1, X_2)}{\sqrt{Q(X_1, X_2)}}X_2 f_w(X_1, X_2)\right]^{2(\lambda+1)}, \tag{5.3.b}$$

$$Q(X_1, X_2) = 1 + 1.464\left\{\mathrm{Min}\left[\frac{B^* X_2}{X_1}, \frac{X_1}{B^* X_2}\right]\right\}^{1.65}, \tag{5.4}$$

$$\begin{aligned} M(X_1, X_2) = {} & \left(1.13 - 0.09B^*\frac{X_2}{X_1}\right) + \left(-0.54 + \frac{0.89}{0.2 + B^* X_2 / X_1}\right)X_2^{\,2} \\ & + \left(0.5 - \frac{1}{0.65 + B^* X_2 / X_1} + 14(1 - B^* X_2 / X_1)^{24}\right)X_2^{\,4}, \end{aligned} \tag{5.5}$$

$$g(X_2) = 1.1 + 0.35X_2^{\,2}, \tag{5.6}$$

$$f_W(X_1, X_2) = \sqrt{\sec\left(\frac{\pi}{2}X_1\sqrt{X_2}\right)}, \tag{5.7}$$

in which we have used the vector notation $\boldsymbol{X} = (X_1, X_2)$. The quantity B^* is defined as

$$B^* = B_2 / B_1, \tag{5.8}$$

which characterizes the shape of the specimen. In these terms, Q is a term due to the elliptical figure of the crack, g, a term due to the backface correction of the specimen's thickness, f_W, a term due to the finite width correction, M, a term called stress intensity magnification factor [13].

It should be noted that the propagating equation (5.2) has a singularity $dX_j / dn \to \infty$ (j = 1,2) on the curve $X_1\sqrt{X_2}$ = 1, which is caused by the finite width correction term $f_W(\boldsymbol{X})$ given by Eqn. (5.7). As the singularity brings a severe problem in the following transformation method, to avoid it, we use the following revised correction term:

$$f_W(\boldsymbol{X}) = \sqrt{\sec\left(\frac{\pi}{2}r_W X_1\sqrt{X_2}\right)}, \tag{5.9a}$$

$$r_W(Y) = \begin{cases} Y & (Y \le y^*) \\ 1-\exp(-c_r Y) & (Y > y^*) \end{cases}, \tag{5.9b}$$

where y^* is a constant which can take on an arbitrary value near 1, and the constant c_r is defined so as to make the correcting function r_W be continuous at the connecting point $Y = y^*$. Equation (5.9) is exactly identical to eqn. (5.7) in $0 < X_1\sqrt{X_2} < y^* < 1$, and does not have any singularity in the region

$$\Omega_X = \{(X_1, X_2) | \, X_1 > 0, X_2 > 0\}. \tag{5.10}$$

Therefore, by the use of eqn. (5.9) instead of eqn. (5.7), the crack growth equation (5.2) is available in Ω_X. As the effective domain of the stress intensity factor is, in the practical situation, limited to a certain sub domain of $\{\boldsymbol{X} | \, 0 \le X_1 \le 1, 0 \le X_2 \le 1\}$, we can make eqn. (5.9) be exactly equivalent to eqn. (5.7) in such a domain by setting y^* sufficiently near to unity.

5.2.2. Equivalent transformation of the basic equation

Using the crack growth equation (5.2) is too difficult to be extended to a stochastic differential equation describing random crack propagation, which is caused mainly by the so-called cross effect between the surface length X_1 and the crack depth X_2. To avoid this difficulty, we propose the following transformation method.

Eliminating the time variable n from eqn. (5.2), we can obtain the following differential equation:

$$\frac{dX_1}{dX_2} = \frac{B^{*\lambda+2}}{\alpha}\left\{\sqrt{X_2 / X_1} g(X_2)\right\}^{2(\lambda+1)} \tag{5.11}$$

The solution of eqn. (5.11) under the initial condition $\boldsymbol{X}(0) = \boldsymbol{x}_0 = (x_{01}, x_{02})$ is analytically given as follows.

$$T(X : x_0) \equiv \frac{1}{\lambda+2}\left(X_1^{\lambda+2} - x_{01}^{\lambda+2}\right) - \frac{B^{*\lambda+2}}{\alpha}\int_{x_{02}}^{X_2} x^{\lambda+1} g(x)dx = 0, \tag{5.12}$$

where

$$\alpha \equiv \varepsilon_2 / \varepsilon_1, \tag{5.13}$$

represents the ratio of the propagating resistances. Equation (5.12) represents the relationship between the surface length $X_1(n)$ and the length depth $X_2(n)$ during the temporal growth of surface cracks. We call it the *trajectory* of the crack growth process [10,14].

Solving eqn. (5.12) with respect to X_1 and X_2 as

$$X_1 = T_1(X_2; x_0), \quad X_2 = T_2(X_1; x_0), \tag{5.14}$$

and again substituting it into eqn. (5.2), we can obtain the following new equation for the

surface crack growth:

$$\frac{dX_j}{dn} = \varepsilon_j Z_n^{2(\lambda+1)} \tilde{g}_j(X_j; x_0) \quad (j = 1,2), \tag{5.15}$$

$$\tilde{g}_1(X_1) \equiv g_1(X_1, T_2(X_1; x_0)), \quad \tilde{g}_2(X_2) \equiv g_1(X_2, T_1(X_2; x_0), X_2) \tag{5.16}$$

Equation (5.15) is different from eqn. (5.2) in the point that the cross effect is eliminated, that is, the two dependent variables X_1 and X_2 are separated in the equation. How ever, it is essentially equivalent to eqn. (5.2) in the point that it gives the same trajectory provided that the same initial condition is given. In this paper, we use eqn. (5.15) as a basic differential equation describing the surface crack growth.

5.3 Probability distribution of the crack propagation

Let us consider the case in which both the loading and the materials' property are random. In this case, ε_j and Z_n must be mathematically treated as stochastic processes. For convenience sake in analysis, we introduce $C_{j,n}$ as

$$\varepsilon_j = \varepsilon C_{j,n} \quad (j = 1,2), \tag{5.17}$$

and set ε as constant. The quantity $C_{j,n}$ represents a non-dimensional random propagating resistance at the point P_j in Fig.5.1 after n cycles of loading. Substituting eqn. (5.17) into eqn. (5.15), we obtain

$$\frac{dX_j}{dn} = \varepsilon C_{j,n} Z_n^{2(\lambda+1)} \tilde{g}_j(X_j; x_0) \quad (j = 1,2), \tag{5.18}$$

We assume that eqn. (5.18) holds as a system of stochastic differential equations under the condition that $C_{j,n}$ and Z_n show temporally random variations. For a probabilistic analysis of the stochastic differential equation (5.18), we confine ourselves to the situation in which the processes $C_{j,n}$ and Z_n are stationary, or at least locally stationary [7].

Let $W(x, n|x_0)$ be the probability distribution function of the solution processes $\boldsymbol{X}(n)$ of eqn. (5.18) under the initial condition $\boldsymbol{X}(0) = \boldsymbol{x}_0$, that is,

$$W(x, n|x_0) = \Pr[X_1(n) \le x_1, X_2(n) \le x_2 \,|\, \boldsymbol{X}(0) = \boldsymbol{x}_0], \tag{5.19}$$

and let $w(x, n|x_0)$ be its density, that is,

$$w(x, n|x_0) = \frac{\partial^2}{\partial x_1 \partial x_2} W(x, n|x_0). \tag{5.20}$$

Applying the Markov approximation method [7], we can obtain the generalized Fokker-Planck equation, which is a partial differential equation to describe the temporal variation of the density w as follows:

$$\frac{\partial}{\partial x}w(x,n|x_0) = -\sum_{j=1}^{2}\beta_j(n)\frac{\partial}{\partial x_j}\{\tilde{g}_j(x_j;x_0)w(x,n|x_0)\}$$
$$-\sum_{j=1}^{2}\gamma_{j,j}(n)\frac{\partial}{\partial x_j}\left\{\tilde{g}_j(x_j;x_0)\frac{d\tilde{g}_j(x_j;x_0)}{dx_j}w(x,n|x_0)\right\} \tag{5.21}$$
$$+\sum_{j=1}^{2}\sum_{k=1}^{2}\gamma_{j,k}(n)\frac{\partial}{\partial x_j\partial x_k}\{\tilde{g}_j(x_j;x_0)\tilde{g}_k(x_k;x_0)w(x,n|x_0)\},$$

$$\beta_j(n) = \varepsilon E\left[C_{j,n}Z_n^{2(\lambda+1)}\right] \quad (j=1,2), \tag{5.22}$$

$$\gamma_{j,k} = \varepsilon^2\int_{-\infty}^{0}\boldsymbol{K}\left[C_{j,n}Z_n^{2(\lambda+1)}, C_{j,n+n'}Z_{n+n'}^{2(\lambda+1)}\right]dn' \quad (j=1,2), \tag{5.23}$$

where $E[\ ]$ represents an operator to take expectation of random variable, and $\boldsymbol{K}[\ ]$ is defined as

$$\boldsymbol{K}[A,B] = E[AB] - E[A]E[B], \tag{5.24}$$

which is an operator to take covariance.

According to Ref.[8], the exact and analytical solution of eqn. (5.21) can be given as follows (see Appendix):

$$w(x,n|x_0) = \frac{1}{\tilde{g}_1(x_1)\tilde{g}_2(x_2)4\pi\sqrt{G(n)}} \times \exp\left\{-\frac{G_{22}x_1^{*2} - (G_{12}+G_{21})x_1^*x_2^* + G_{11}x_2^{*2}}{4G(n)}\right\}, \tag{5.25}$$

$$G(n) = G_{11}(n)G_{22}(n) - \{G_{12}(n)+G_{21}(n)\}^2/4, \tag{5.26}$$

$$G_{jk}(n) = \varepsilon^2\int_0^n dn'\int_{-\infty}^{0}\mathrm{K}[C_{j,n'}Z_{n'}^{2(\lambda+1)}, C_{k,n'+n''}Z_{n'+n''}^{2(\lambda+1)}]dn'' \quad (j,k=1,2), \tag{5.27}$$

$$x_j^*(x_j,n;x_0) = \int_{x_{0j}}^{x_j}\frac{dx}{\tilde{g}(x,x_0)} - \varepsilon E[C_{j,n}Z_n^{2(\lambda+1)}]n \quad (j=1,2), \tag{5.28}$$

It should be noted that, if $\lambda > 0$, the so-called normalizing condition is not satisfied with respect to the density $w(x,n|x_0)$ in Ω_X, that is, for $n > 0$,

$$\iint_{\Omega_X} w(x,n|x_0)dx < 1. \tag{5.29}$$

This inequality means that the solution processes become absorbed one after another into the death point $\mathcal{B}$ such that

$$\mathcal{B} = \{(X_1,X_2)|X_1 \to \infty \text{ or } X_2 \to \infty\}, \tag{5.30}$$

as time elapses [7]. The quantity defined as

$$P_{\mathcal{B}}(n|x_0) = 1 - \iint_{\Omega_X} w(x,n|x_0)dx_0, \tag{5.31}$$

represents the probability that the solution process is absorbed into the death point $\mathcal{B}$ after n cycles.

5.4. Application to probabilistic LBB assessment

In this section, by utilizing the probability distribution of the surface crack propagation, we discuss a way of calculating the probability that the LBB holds in piping systems. To this end, as mentioned in Section 5.2, we suppose that our model is available as a model of random surface crack growth in piping, that is, e suppose a pipe with the internal radius B_1/π and the thickness B_2.

Let $g_L(X)$ and $g_B(X)$ be the limit state function for leak and that for break in the X plane, respectively, which means that the leak of internal fluid takes place when the solution process $X(n)$ arrives at the region $\{X|g_L(X) \le 0\}$ for the first time, and that the break of the component takes place in the region $\{X|g_B(X) \le 0\}$.

As for the leak occurrence, the following fact is generally recognized. When the crack depth grows to 70-80% of the specimen's thickness, the material shows a large scale yielding around the deepest point of the crack, and, after that, the crack rapidly grows, changing its shape from an elliptical figure, to penetrate the thickness. That is to say, the required time from the beginning of the large scale yielding to the penetration of the thickness is sufficiently small compared with the total residual life of the specimen. Hence, we can make a conservative assessment of the specimen's residual life by neglecting this time, in other words, by assuming that the leak takes place when the crack depth grows to 70-80% of the thickness. According to this fact, we can express the function $g_L(X)$ as

$$g_L(X) = x_{c2} - X_2, \tag{5.32}$$

where x_{c2} is a constant taking on about 0.7-0.8, which may be interpreted a limit of the non-dimensional depth X_2 that the linear fracture mechanics is applicable. The domain Ω_X given by Eqn. (5.10) is decomposed into the following three sub domains:

$$\begin{cases}\Omega_{\mathrm{SAFE}} = \{X|g_L(X) > 0, g_B(X) > 0\} & \cdots\cdots \text{Safe} \\ \Omega_{\mathrm{LBB}} = \{X|g_L(X) \le 0, g_B(X) > 0\} & \cdots\cdots \text{Leak Before Break} \\ \Omega_{\mathrm{B}} = \{X|g_B(X) \le 0\} & \cdots\cdots \text{Break}\end{cases} \tag{5.33}$$

which are schematically illustrated in fig.5.3.

All the solution processes finally arrive at the break zone Ω_B, and they are absorbed in to the death point $\mathcal{B}$ in the long run. However, if it passes Ω_{LBB} before arriving at Ω_B, we can avoid a catastrophic failure of the piping system by stopping the operation of the system after the leak of the internal fluid. Otherwise, the break of the pipe takes place before we detect the surface crack growth, and it brings a catastrophic failure such as the so-called guillotine break. Therefore, in designing a piping system, we need to obtain the knowledge on the probability that the solution process passes the

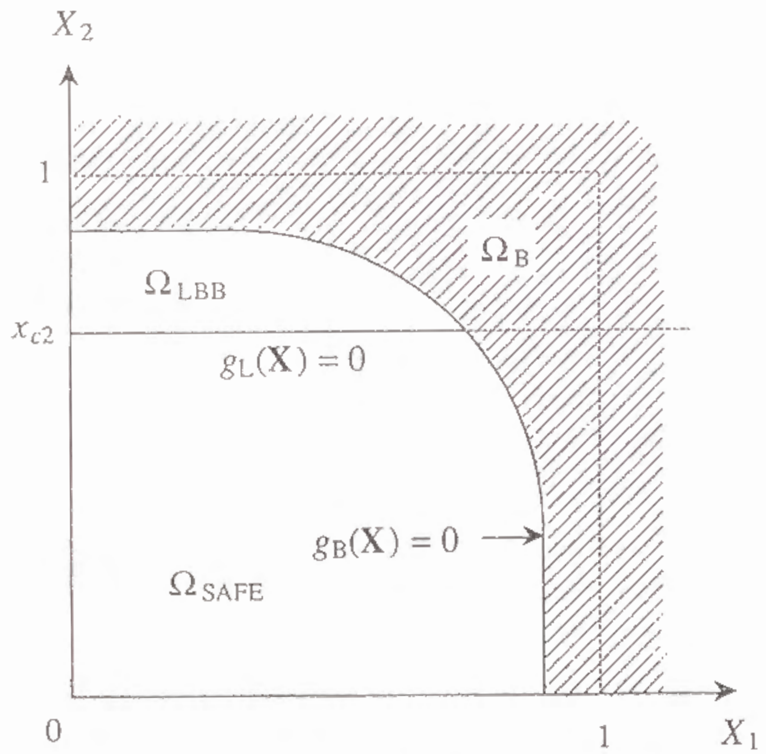

Fig. 5.3. Schematic illustration of limit state functions in X-plane.

domain Ω_{LBB}.

To derive this probability, we introduce the following three quantities:

$$H_{\mathrm{SAFE}}(n) = \Pr[X(n) \text{ lies in } \Omega_{\mathrm{SAFE}}], \tag{5.33a}$$

$$H_{\mathrm{LBB}}(n) = \Pr[X(\cdot) \text{ passes } \Omega_{\mathrm{LBB}} \text{ within } n \text{ cycles}], \tag{5.33a}$$

$$H_{\mathrm{B}}(n) = \Pr[X(\cdot) \text{ arrives at } \Omega_{\mathrm{B}} \text{ within } n \text{ cycles without passing } \Omega_{\mathrm{LBB}}]. \tag{5.33a}$$

Then, the following normalizing relationship holds:

$$H_{\mathrm{SAFE}}(n) + H_{\mathrm{LBB}}(n) + H_{\mathrm{B}}(n) = 1. \tag{5.34}$$

Making an infinitesimal increment of $H_{\mathrm{LBB}}(n)$, we have

$$\begin{aligned}\Delta H_{\mathrm{LBB}}(n) &= H_{\mathrm{LBB}}(n+\Delta n) - H_{\mathrm{LBB}}(n)\\ &= \Pr[X(\cdot) \text{ passes } \Omega_{\mathrm{LBB}} \text{ for the first time in the time interval } (n, n+\Delta n)]\\ &= r_{\mathrm{LBB}}(n)\Delta n\, H_{\mathrm{SAFE}}(n),\end{aligned} \tag{5.35}$$

where $r_{\mathrm{LBB}}(n)$ is the following LBB occurring rate:

$$r_{\mathrm{LBB}}(n)\Delta n = \Pr[X(\cdot) \text{ arrives at } \Omega_{\mathrm{LBB}} \text{ in } (n, n+\Delta n) | X(\cdot) \in \Omega_{\mathrm{SAFE}}], \tag{5.36}$$

which is given by the following equation:

$$r_{\mathrm{LBB}}(n) = \lim_{\Delta n \to 0} \frac{1}{\Delta n} \iint_{\Omega_{\mathrm{LBB}}} dx \iint_{\Omega_{\mathrm{SAFE}}} w(x, n+\Delta n | x', n) w^*(x', n | x_0) dx', \tag{5.37}$$

The function $w(x, n+\Delta n\, x', n)$ represents the probability density function of $X(n+\Delta n)$ under the condition that $X(n)$ takes on x', which is easily derived by shifting the time origin in the discussion of the previous section, and it is identical to $w(x, \Delta n\, x')$ because of the homogeneous property of the stochastic process $X'(n)$. The function $w^*(x, n|x_0)$ represents the probability density function of $X(n)$ under the condition that $X(n)$ lies in Ω_{SAFE}. By neglecting the probability that the crack length decreases, $w^*(x, n|x_0)$ is approximately given as follows:

$$w^*(x, n|x_0) = \frac{w(x, n|x_0)}{\iint_{\Omega_{\mathrm{SAFE}}} w(x, n|x_0) dx} \quad (x \in \Omega_{\mathrm{SAFE}}). \tag{5.38}$$

Substituting eqn. (5.38) into eqn. (5.37), we can obtain the following expression for $r_{\mathrm{LBB}}(n)$:

$$r_{\mathrm{LBB}}(n) = \frac{1}{\iint_{\Omega_{\mathrm{SAFE}}} w(x, n|x_0) dx} \iint_{\Omega_{\mathrm{LBB}}} dx \iint_{\Omega_{\mathrm{SAFE}}} q(x|x') w(x', n|x_0) dx', \tag{5.39}$$

where

$$q(x|x') \equiv \lim_{n \to 0} \frac{\partial}{\partial n} w(x, n|x'). \tag{5.40}$$

We should note that this limiting procedure is formal and in the numerical calculation we have to use the value corresponding to an adequately small n.

On the other hand, by neglecting the probability that the crack length decreases, $H_{\mathrm{SAFE}}(n)$ defined by eqn. (5.33c) is approximately given by

$$H_{\mathrm{SAFE}}(n) = \iint_{\Omega_{\mathrm{SAFE}}} w(x,n|x_0)dx. \tag{5.41}$$

Substituting eqns. (5.39) and (5.41) into eqn. (5.35), we obtain the following differential equation for $H_{\mathrm{LBB}}(n)$:

$$\frac{d}{dn}H_{\mathrm{LBB}}(n) = \iint_{\Omega_{\mathrm{LBB}}} dx \iint_{\Omega_{\mathrm{SAFE}}} q(x|x')w(x',n|x_0)dx'. \tag{5.42}$$

As neither leak nor break takes place in the initial time $n = 0$, the initial condition for eqn. (5.42) is $H_{\mathrm{LBB}}(0) = 0$. Therefore, by integrating eqn. (5.42), we finally obtain the following expression for $H_{\mathrm{LBB}}(n)$:

$$H_{\mathrm{LBB}}(n) = \int_0^n dn' \iint_{\Omega_{\mathrm{LBB}}} dx \iint_{\Omega_{\mathrm{SAFE}}} q(x|x')w(x',n|x_0)dx'. \tag{5.43}$$

The quantity $\lim_{n\to\infty} H_{\mathrm{LBB}}(n)$ gives the probability that the LBB finally holds.

As already mentioned, we can avoid a catastrophic failure of the piping system if the solution process $X(\cdot)$ passes Ω_{LBB} before reaching the break zone, provided that the ability of the leak monitor is reliable. Therefore,

$$R(n) = H_{\mathrm{SAFE}}(n) + H_{\mathrm{LBB}}(n) = 1 - H_{\mathrm{B}}(n), \tag{5.44}$$

gives a kind of measure of the reliability of the piping system.

5.5. Concluding remarks

In this chapter, based upon the stochastic fracture mechanics, we have derived the probability distribution of random surface crack propagation in a closed form by utilizing the Markov approximation method [7], and with the aid of it, discussed the way how to calculate the probability of LBB in piping systems. As the obtained results are mathematically expressed in closed forms, our approach has the following two advantages:

(i) Some physical parameters are analytically included in our results, and thus we can easily take into account the uncertainties associated with them. For example, the randomness of the initial state vector $x_0 = (x_{01}, x_{02})$ can be easily introduced in to our model [1].

(ii) We can evaluate an extremely high reliability by the use of our results, and therefore, in addition to the simulation technique, this method may be an alternative useful technique in the probabilistic LBB assessment. Since extremely high reliability is generally needed in reactor technology, the advantage (ii) has an important meaning to apply our results to practical safety assessment in piping systems.

To assess the probability of LBB numerically, we have to evaluate multi-fold integration appearing in eqn. (5.39) numerically with high accuracy. Therefore, we need

to develop an efficient computing routine to do this. In addition to this point, we have to formulate a more precise condition for LBB by taking the effect of crack opening displacement, etc. into account. These are future problems.

5.6 Appendix: Derivation of eqn. (5.25)

Here we summarize the procedure for solving the generalized Fokker-Planck equation (5.21) according to Ref.[8].

First, we transform the dependent variable w in eqn. (5.21) into v as

$$v(x_1,x_2,n) = \tilde{g}_1(x_1;x_0)\tilde{g}_2(x_2;x_0)w(x_1,x_2,n), \tag{5.A.1}$$

and the two space variables x_1 and x_2 into ζ_1 and ζ_2 as

$$\zeta_j = \int_{x_{0,j}}^{x_j} \frac{dx_j'}{\tilde{g}_j(x_j';x_0)} - \int_0^n \beta_j(n')dn' \quad (j = 1,2), \tag{5.A.2}$$

Then, eqn. (5.21) is transformed into the following diffusion equation of no-diagonal type:

$$\frac{\partial v}{\partial n} = \sum_{j=1}^{2}\sum_{k=1}^{2} \gamma_{jk}(n)\frac{\partial^2 v}{\partial\zeta_j \partial\zeta_k}, \tag{5.A.3}$$

To solve eqn. (5.A.3), we introduce the Fourier transformation,

$$\hat{v}(\eta_1,\eta_2,n) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i\eta_1\zeta_1 - i\eta_2\zeta_2} v(\zeta_1,\zeta_2,n)d\zeta_1 d\zeta_2, \tag{5.A.4}$$

instead of $v(\eta_1,\eta_2,n)$, in which $i = \sqrt{-1}$ and we set $v = 0$ except for the range:

$$\zeta_j \le \int_{x_{0,j}}^{\infty} \frac{dx_j}{\tilde{g}_j(x_j;x_0)} - \int_0^n \beta_j(n')dn' \quad (j = 1,2). \tag{5.A.5}$$

The differential equation for $\hat{v}$ corresponding to eqn. (5.A.3) takes the following form:

$$\frac{\partial \hat{v}}{\partial n} = \sum_{j=1}^{2}\sum_{k=1}^{2} \gamma_{jk}(n)\eta_j\eta_k\hat{v}. \tag{5.A.6}$$

The initial condition considered here is that $X(0) = x_0$ holds with probability 1, which implies that the initial condition for $w(x_1,x_2,n)$ can be expressed as

$$\lim_{n\to 0} w(x_1,x_2,n) = \delta(x_1 - x_{01})\delta(x_2 - x_{02}). \tag{5.A.7}$$

Hence, the initial condition for $\hat{v}$ is, by the transforming eqns. (5.A.1) and (5.A.2), given as follows:

$$\lim_{n\to 0} \hat{v}(\eta_1,\eta_2,n) = 1. \tag{5.A.8}$$

Using this condition, we can easily integrate eqn. (5.A.6) as follows:

$$\hat{v}(\eta_1,\eta_2,n) = \exp\left\{-\sum_{j=1}^{2}\sum_{k=1}^{2} G_{jk}(n)\eta_j\eta_k\right\}, \tag{5.A.9}$$

in which $G_{jk}(n)$ is defined by eqn. (5.27). The solution of eqn. (5.A.3) can be obtained by inverse Fourier transformation procedure form eqn. (5.A.10), that is,

$$v(\zeta_1,\zeta_2,n) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\zeta_1\eta_1+i\zeta_2\eta_2}\hat{v}(\eta_1,\eta_2,n)d\eta_1 d\eta_2 . \tag{5.A.10}$$

Substituting eqn. (5.A.9) into eqn. (5.A.10), carrying out the integration, and inversely transforming back to the original variable according to eqns. (5.A.1) and (5.A.2), we can finally obtain the solution of the generalized Fokker-Planck equation (5.21) as eqns.(25) ~ (28).

References

[1] H. Tanaka and A. Tsurui, Reliability degradation of structural components in the process of fatigue crack propagation under stationary random loading, *Engrg. Frac. Mech.* **27** (1987) 501-516.

[2] T. Fujioka, K. Kashima, Y. Takahashi, An application study of methodologies of probabilistic fracture mechanics aimed at reliability assessment of nuclear components, *Komae Research Laboratory Rep.* No. T88080 (1989) (in Japanese).

[3] D.O. Harris, E.Y. Lim and D.D. Dedhia, Probability of pipe fracture in the primary coolant loop of a PWR plant, Volume 5: Probabilistic fracture mechanics analysis, NUREG/CR-2189, Vol. 5, (1981) pp. 43-49.

[4] W. Marshall, An assessment of the integrity of PWR pressure vessels, available from H. M. Stationary Office, London (1976).

[5] T.Y. Lo, R.W. Mensing, H.H. Woo and G.S. Holman, Probability of pipe failure in the primary coolant loops of combustion engineering PWR plants, Volume 2: Pipe failure induced by crack growth, NUREG/CR-3663, Vol.**2** (1984) p.29.

[6] H.H. Woo, R.W. Mensing and B.J. Benda, Probability of pipe failure in the primary coolant loops of Westinghouse PWR plants, Volume 2: Pipe failure induced by crack growth, NUREG/CR-3663, Vol.**2** (1984) p.20.

[7] A. Tsurui and H. Ishikawa, Application of Fokker-Planck equation to a stochastic fatigue crack growth model, *Structural Safety* **4** (1986) 15-29.

[8] H. Tanaka and A. Tsurui, Random propagation of a semi-elliptical surface crack as a bivariate stochastic process, *Engrg. Frac. Mech.* **33** No.5 (1989) 787-800.

[9] J. Nienstedt, A. Tsurui, H. Tanaka and G.I. Schuëller, Time variant structural reliability analysis using bivariate diffusive crack growth models, *Int. J. of Fatigue* **12** (1989) 83-89.

[10] H. Tanaka, Stochastic properties of semi-elliptical surface crack propagation based upon Newman-Raju's K-expression, *Engrg. Frac. Mech.* **34**, No.1 (1989) 189-200.

[11] J.C. Newman, Jr. and I.S. Raju, Analysis of surface cracks in finite plates under tension or bending loads, NASA TP-1578 (1978) **4**, No.3 (1989) 120.

[12] P.C. Paris and F. Erdogan, A critical analysis of crack propagation laws, *Trans. of ASME*, Ser. D, Vol. **85** (1963) 528-534.

[13] H. Nisitani and Y. Murakami, Stress intensity factor of an elliptical crack or a semi-elliptical crack subjected to tension, *Int. J. of Fracture* **10** (1974) 353-368.

[14] H. Tanaka and A. Tsurui, Proc. of 10th Symposium on Reliability Engineering in Design (JSMS), 173-178 (in Japanese).

Chapter VI

Reliability Analysis of Structural Components under Fatigue Environment Including Random Overloads

6.1. Introduction

We often come across a situation in which the non-linearity in fatigue crack propagation processes cannot be disregarded. For instance, if the loading process is "strongly" random, there will be a possibility of "overloads" that causes the retardation effect in the crack propagation. In this case, we can not directly apply the Paris' law to describe the crack propagation. On the other hand, if overload appears, we must take instantaneous ductile failure into reliability analysis. Thus, the so-called two-criteria approach is required, which describe the fatigue and ductile failure modes.

In this chapter, supposing a situation in which a structural component suffers from random cyclic loads including overloads, we investigate the behavior of its hazard rate and reliability function. A concept of "delay time" is first introduced according to Arone's study [1,2], for the purpose to derive the distribution of the residual life of the component up to its failure. Making use of the probability distribution function of the delay time including the retardation effect due to overloads, we then revise a probability distribution function of the residual life of the component, which is previously obtained through Markov approximation method by Tsurui *et al.* [3,4], so as to reflect the retardation effect. Using this result, a distribution function of crack length, which reflects the retardation effect, can be obtained as an approximation.

In order to take the ductile failure mode into account, we have to construct a limit state function reflecting the two failure modes simultaneously. For this purpose, Burdekin-Stone's failure criteria [5] is useful to describe the ductile and fatigue failure. This criteria asserts that the component fails by at least one of following two events, (i) the loading stress exceeds a critical value which depends on the current crack length, and, (ii) the fatigue crack length exceeds a critical value.

Since the limit state function based upon the Burdekin-Stone's criteria is not of simple form to calculate the hazard rate analytically, so the importance sampling simulation technique, which is a kind of Monte-Carlo simulation, is applied for the evaluation of the hazard rate [6]. Making use of this simulation method, an approximate value of the hazard rate is evaluated with high accuracy in comparatively short CPU time. As the result, the reliability function can be calculated from the hazard rate, and it is observed that the behavior of the reliability function reflects the two failure modes above. Interesting results are that the retardation due to overloads has almost no influence in the high reliability range, and that the ductile failure mode is rather dominant over the first half of the life of components.

5.2 Probability distribution of total delay time

In this section, we simply review the Arone's method to derive the probability distribution of total delay time caused by random overloading, and partly extend it to more general form.[1,2]

Here we assume that the component is loaded with two distinct loadings. One is a constant amplitude cyclic loading, and the other the overloads which occur randomly and independently from the cyclic loading. Figure 6.1 shows the schematic illustration of superimposed overload and retardation of crack growth rate.

In the figure, $X_i(n)$ represents the crack length under the retardation effect caused by the overload occurred at the n_i-th cycle, and $\tilde{X}(n)$ a virtual crack length under the condition of no overloads. Let us define a delay time $n_{d_i}(y)$ at a crack length y as

$$n_{d_i}(y) = X^{-1}_i(y) - \tilde{X}^{-1}_i(y). \quad (6.1)$$

Since the effect of the overload vanishes as time elapses, $n_{d_i}(y)$ converges to a constant value in the limit of $y \to \infty$. We call this limiting value, $n_{d_i} = \lim_{y \to \infty} n_{d_i}(y)$, "delay time" due to the overload occurred at the n_i-th cycle.

If plural overloads occur, the total delay time does not become a simple sum of

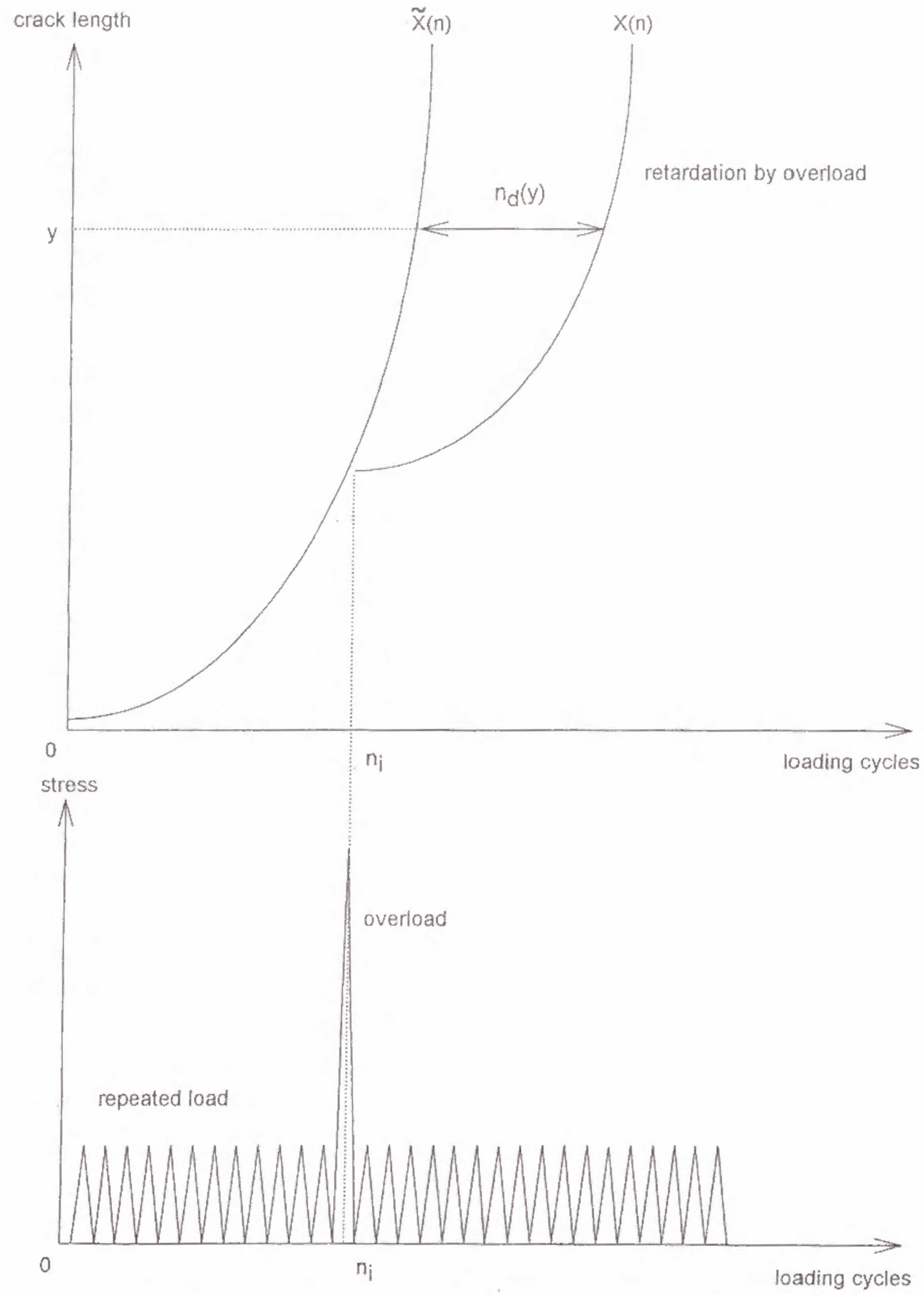

Figure 6.1: The schematic illustration of superimposed overload and retardation of crack growth.

each delay time generally, owing to the cross-effect. However, if the structural component is suitably designed, the occurrences of overloading can be considered as rare events. Therefore, we assume that the total delay time $N_d(n)$ after n cycles of loading is approximately given as

$$N_d(n) = \sum_i n_{d_i}. \tag{6.2}$$

Generally, the delay time as well as the occurrence times of the overloads cannot be specified from the deterministic approach and is not always expressed by theoretical expression, so that it is suitable idea that the total delay time, $N_d(n)$, has a stochastic nature due to the randomness associated with the number and the magnitude of overloads. Hence, we treat the total delay time as a stochastic process. The probability distribution function of total delay time is expressed as

$$F(n_d, n) = \Pr[N_d(n) \le n_d]. \tag{6.3}$$

In order to determine $F(n_d, n)$, we introduce the following transition probability distribution:

$$P(n_d | n'_d, n) = \Pr[N_d(n^{(+)}) \le n_d | \text{Overload occurs at the } n\text{-th cycle, } N_d(n) = n'_d], \tag{6.4}$$

where $N_d(n^{(+)})$ represents the total delay time after n cycles of loading, including the delay time due to the overload at the n-th cycle. Letting $q(n)$ be a probability that an overload occurs at the n-th cycle. Classifying the n-th load into overload and no overload, we can derive the following equation:

$$F(n_d, n+1) = \{1 - q(n)\}F(n_d, n) + q(n)\int_0^{n_d} P(n_d | n'_d, n) d_{n'_d} F(n'_d, n), \tag{6.5}$$

in which the integral in the second term of the right hand of the equation is of Stieltjes type. If the scale of n is set appropriate, the above equation is rewritten as the following differential integral equation:

$$\frac{\partial F(n_d, n)}{\partial n} = -q(n)F(n_d, n) + q(n)\int_0^{n_d} P(n_d | n'_d, n) d_{n'_d} F(n'_d, n), \tag{6.6}$$

where $q(n)$ is re-interpreted as a probability that an overload occurs in a unit time. The initial and boundary conditions for $F(t_d, t)$ are as follows:

(a) No overloads occur at the initial time, that is,

$$\lim_{t \to 0} F(t_d, t) = \begin{cases} 0 & (t_d \le 0), \\ 1 & (t_d > 0). \end{cases} \tag{6.7}$$

(b) The total delay time cannot be negative, and cannot exceed the elapsed time, that is,

$$\lim_{t_d \to 0} F(t_d, t) = 0, \quad \lim_{t_d \to t} F(t_d, t) = 1. \tag{6.8}$$

If the transition probability $P(n_d | n'_d, n)$ is known, then we can obtain the probability distribution function of the total delay time $F(n_d, n)$ as the solution of eqn. (6.6) under

the initial condition of eqn. (6.7) and the boundary condition of eqn. (6.8).

The transition probability $P(n_d | n'_d, n)$ is determined from the probability distribution function of the delay time for single overload. Let $\nu_d(t)$ be a delay time due to the overload at the n-th cycle, then the following equation holds:

$$N_d(n^{(+)}) = n'_d + \nu_d(t). \tag{6.9}$$

The transition probability can be expressed as

$$P(n_d | n'_d, n) = \Pr[n'_d + \nu_d(n) \le n_d] = \Pr[\nu_d(n) \le n_d - n'_d], \tag{6.10}$$

which means that we can determine the transition probability $P(n_d | n'_d, n)$ if the probability of the delay time $\nu_d(t)$ is known.

Arone has proposed a certain expression for $P(n_d | n'_d, n)$ [2], under the assumptions that (i) the loading process is a constant amplitude process except overloads, and (ii) a parameter associated with the magnitude of retardation effect obeys an exponential distribution. In this paper, in order to generalize the shape of the distribution and to clarify its influence, Weibull distribution is applied instead of exponential distribution, and the following transition probability distribution function is used:

$$P(n_d | n'_d, n) = \begin{cases} 0 & (n_d < n'_d), \\ 1 & (n_d \ge n'_d \text{ and } n_\infty + n'_d - n \le 0), \\ 1 - \exp\left[-\left\{\dfrac{n_d - n'_d}{\xi(n_\infty + n'_d - n)}\right\}^{\rho}\right] & (\text{otherwise}), \end{cases} \tag{6.11}$$

in which ρ and ξ represent a shape and a scale parameters of the Weibull distribution, respectively. The quantity n_∞ represents the number of cycles when the crack with initial length x_0 grows to infinity under the condition of no overloads. If the crack propagation obeys Paris' law, n_∞ is evaluated with

$$n_\infty = \frac{x_0^{-\lambda}}{\lambda \varepsilon Z_0^{2(\lambda+1)}}, \tag{6.12}$$

where ε, Z_0, $2(\lambda+1)$ represent a crack propagation resistance, an amplitude of loading process, an exponent of the Paris's law, respectively. It should be noted that eqn. (6.11) reduces to a deterministic distribution in the limit of $\rho \to \infty$, which corresponds to the case in which the occurrence of overloads is random but its magnitude is constant.

In this paper, we apply the transition probability of total delay time, $P(n_d | n'_d, n)$, to the random fatigue crack propagation process with random propagation resistance and random loading amplitude. Then, the parameter n_∞ is fixed approximately through setting ε and Z_0 to the expectation value of random propagation resistance and of random loading amplitude, respectively.

6.3 Residual life distribution including retardation effect

Under the condition of no overloads, we make an assumption that the fatigue crack propagates in accordance with Paris law. Then, the behavior of the crack length obeys the following differential equation:

$$\frac{dX(n)}{dn} = \varepsilon C_n Z_n^{2(\lambda+1)} X^{\lambda+1}, \tag{6.13}$$

in which $X(n)$, Z_n, C_n, ε, and $2(\lambda+1)$ represent a crack length after n cycles of loading, an loading amplitude at the n-th cycle, a normalized propagation resistance at the crack tip after n cycles, a mean propagating resistance, and an exponent of Paris's low, respectively. In eqn. (6.13), we also assume that the both processes of Z_n and C_n are locally stationary, and that $\lambda > 0$.

Let x_0 be initial crack length, i.e., $X(0) = x_0$, and N be the number of loading cycles in which the crack length becomes to a pre-specified critical length x_c for the first time. Then the probability distribution function of N,

$$H(n|x_0, x_c) = \Pr[N \le n], \tag{6.14}$$

is called residual life distribution for the crack propagation. If we can assume both correlations of the processes Z_n and C_n to decay exponentially, then the residual life distribution function $H(n|x_0, x_c)$ is given as follows [3,4]:

$$H(n|x_0, x_c) = \Phi\left[\frac{x_0^{-\lambda} - x_c^{-\lambda} - \lambda\varepsilon M_c M_z n}{\lambda\sqrt{G(n)}}\right], \tag{6.15}$$

$$G(n) = \frac{\alpha_0\sigma_c}{2\lambda+1}\left\{x_0^{-(2\lambda+1)} - \left(x_0^{-\lambda} - \lambda\varepsilon M_c M_z \,\mathrm{Min}[n, n_\infty]\right)^{\frac{2\lambda+1}{\lambda}}\right\} + \varepsilon^2 M_c^{\,2}\sigma_c^2 n_z n$$
$$+ \varepsilon^2\sigma_c^2\sigma_z^2 n_z \int_0^{\mathrm{Min}[n,n_\infty]} \frac{\left(x_0^{-\lambda} - \lambda\varepsilon M_c M_z n'\right)^{\frac{2\lambda+1}{\lambda}} dn'}{\left(x_0^{-\lambda} - \lambda\varepsilon M_c M_z n'\right)^{\frac{2\lambda+1}{\lambda}} + \frac{\varepsilon M_c M_z n_z}{\alpha_0}}, \tag{6.16}$$

where M_c, M_z, σ_c^2, σ_z^2 represent the means and the variances of C_n and $Z_n^{2(\lambda+1)}$, respectively. The constants n_z and α_0 are a time duration of correlation cycles of the loading amplitude and a spatial correlation distance of random propagation resistance. The notation $\Phi[\bullet]$ represents a complementary function of the standard normal

distribution

Under the condition that overloads occur, the residual life can be reduced to the total delay time and normal crack growth time. Consequently, the residual life distribution under the condition that the delay time up to n-th cycle equals to n_d is given as $H(n - n_d|x_0, x_c)$, the residual life distribution including the retardation effect, $H_R(n|x_0, x_c)$, is led to

$$H_R(n|x_0, x_c) = \int_0^n H(n - n_d|x_0, x_c)\, d_{n_d} F(n_d, n), \tag{6.17}$$

in which $F(n_d, n)$ is the probability distribution function of the total delay time, as is introduced in the preceding section.

Equation (6.17) is concerned only with the retardation effect and other failure factors are not introduced in the model. This leads the life to safety side, and is insufficient for the life estimation. Overloads sometimes causes not only the retardation of crack propagation, but also the ductile failure, possibly. The failure of the component has a concern in several parameters, such as tensile strength, fracture toughness, loading amplitude, etc. These are often uncertain parameters in their values, so that they are regarded as random variables. Since the condition of failure depends on the crack length as well as these parameters, it is necessary to know the distribution function for the crack length in order to evaluate the hazard rate and failure probability.

Let $W(x, n|x_0)$ be the probability distribution function of the crack length for no retardation, defined by

$$W(x, n|x_0) = \Pr[X(n) \le x \mid X(0) = x_0]. \tag{6.18}$$

Generally, $W(x, n|x_0)$ is connected with the residual life distribution, $H(n|x_0, x_c)$, through the approximate relationship as follows

$$W(x_c, n|x_0) + H(n|x_0, x_c) \approx 1, \tag{6.19}$$

whenever the critical length x_c is fixed, for reason that $X(n)$ is a non-decreasing process. In a strict sense, there exists some decrease of $X(n)$ which is caused by stochastic diffusion in Markov approximation, but its probability is negligible if n is sufficiently large. Equation (6.19) also means that $W(x_c, n|x_0)$ can be approximated to $1 - H(n|x_0, x_c)$, if the parameter x_c is varied in the range $x_c > x_0$.

Since the crack propagation process does not decrease, the relationship in eqn. (6.19) generally holds whether the retardation occurs or not. Now, the residual life distribution function including retardation, $H_R(n|x_0, x_c)$, is expressed in eqn (6.17), so that, conversely, the crack length distribution function including retardation effect, $W_R(x, n|x_0)$, can be approximated to

$$W_R(x, n|x_0) \approx 1 - H_R(n|x_0, x). \tag{6.20}$$

6.4. Hazard rate and reliability function

In the preceding section, the life distribution function has been revised so as to reflect the retardation effect by random overloads under fatigue environment, and, as the result, an approximation has been made for the crack length distribution function of the crack length including overloads.

In this section, the hazard rate is evaluated in consideration of two failure modes, which are fatigue failure mode by normal load and ductile failure mode by overload. For this purpose, it is necessary to specify the limit state function to define the failure domain in the first place. Among the various methods to describe the failure criterion, the two-criteria approach has been selected for this purpose. As is well known, the two-criteria approach is an empirical model that combines the brittle fracture and the ductile failure. Among the various limit state functions as proposed by different authors, the Burdekin-Stone's relation has been selected [5]. On the calculation, following six random variables are introduced as uncertainty factors with relation to failure [6]:

(i) X_1: Current crack length, [mm]
(ii) X_2: Initial crack length, [mm]
(iii) X_3: Loading stress, [N mm^{-2}]
(iv) X_4: Yield strength, [N mm^{-2}]
(v) X_5: Tensile strength, [N mm^{-2}]
(vi) X_6: Fracture toughness, [N mm$^{-1.5}$]

and a vector notation $\mathbf{X} = (X_1, \cdots, X_6)$ are used as well. Some notations of the above variables are redefined as $X_1 = X(n)$, $X_2 = x_0$ and $X_3 = Z_n$.

From the substitution of these variables to Burdekin-Stone's failure criteria, the limit state function leads to

$$g(\mathbf{X}) = \frac{2X_3}{X_4 + X_5} - \frac{K_I}{K_{IC}} \sqrt{\frac{8}{\pi^2} \ln \sec\left(\frac{\pi}{2} \frac{2X_3}{X_4 + X_5}\right)}, \tag{6.21}$$

where K_I stands for the current stress intensity factor and K_{IC} the critical stress intensity factor. The failure domain corresponds to the negative side of $g(\mathbf{X})$. As is usual, K_I is expressed as

$$K_I = X_1 \sqrt{\pi X_3}, \tag{6.22}$$

which can imply the geometry factor, if necessary. Usually, K_{IC} is the same as X_6 if the crack length is neither too small nor too large in comparison to the component size. In order to describe the linear elastic failure mode in the extra region, K_{IC} is empirically expressed by Feddersen scheme [8] as a function of X_2, X_4, X_5 and X_6.

If the distribution of the random variables is specified, the failure probability is calculated from the integral in the failure domain, but it evaluates an instantaneous value at the n-th cycle. This means that the value of the integral produces the hazard rate, that is, the conditional probability of failure at the n-th cycle under the condition of no failure before the n-th cycle.

Under the circumstances in the previous section, it is noted that the random variable X_3 obeys the distinctive distribution on the occasion of normal load and overload, respectively, while both play the same role in eqn. (6.21). Therefore, the hazard rate is separated into two conditional hazard rates, $h^{(N)}(n)$ for normal load and $h^{(O)}(n)$ for overload. The former corresponds to the hazard rate under the condition that a normal-load takes place at the n-th cycle and the latter that an overload takes place at the n-th cycle. Each conditional hazard rate can be evaluated with the conditional probability density, that is,

$$h^{(N)}(n) = \int_{g(\mathbf{X})\le 0} f_X(\mathbf{x}, n | X_3 \text{ is a normal load}) d\mathbf{x}, \tag{6.23}$$

$$h^{(O)}(n) = \int_{g(\mathbf{X})\le 0} f_X(\mathbf{x}, n | X_3 \text{ is an overload}) d\mathbf{x}, \tag{6.24}$$

respectively, where $f_X(\mathbf{x}, n)$ is the joint probability density function for $\mathbf{X}$.
Since the occurrence probabilities for overload and normal load are $q(n)$ and $1 - q(n)$, respectively, the total hazard rate $h(n)$ can be obtained through

$$h(n) = \{1 - q(n)\} h^{(N)}(n) + q(n) h^{(O)}(n). \tag{6.25}$$

On the assumption that the random variables are independent, the joint probability density function is reduced to

$$f_X(\mathbf{x}, n|\bullet) = w^*(x_1, n|x_2) f_{X_2}(x_2) f_{X_3}(x_3|\bullet) f_{X_4X_5}(x_4, x_5) f_{X_6}(x_6), \tag{6.26}$$

where $w^*(x_1, n|x_2)$ is the normalized probability density function for crack length,

$$w^*(x_1, n|x_2) = \frac{\dfrac{\partial}{\partial x} W_R(x_1, n|x_2)}{W_R(x_c, n|x_2)}. \tag{6.27}$$

This normalization assures the condition that the failure does not occur up to the n-th cycle.

Once the hazard rate is known for arbitrary n, the reliability function is easily derived. Generally, the reliability function $R(n)$ is connected with the hazard rate through

$$h(n) = -\frac{1}{R(n)} \frac{dR}{dn}. \tag{6.28}$$

Hence, $R(n)$ is expressed by

$$R(n) = \exp\left(-\int_0^n h(n) dn\right). \tag{6.29}$$

6.5. Numerical studies

The integrals in eqns. (6.23) and (6.24) are too complicated to be analytically carried out. So, the failure analysis is performed with the numerical integration with the aid of the importance sampling simulation as a kind of Monte-Carlo simulation technique, in particular, the software package ISPUD [7].

Table 6.1. shows the type of distribution and its parameters of each random variable used in the calculation. The material parameters are specified from material data of the high strength steel. Since the coefficients of variation of X_4 and X_5 are both small in this example, so X_4 and X_5 are assumed to be independent of each other.
The parameters to specify the distribution of X_1 are shown in table 6.2.

Needless to say, M_Z and σ_z must be calculated from the distribution of X_3 (normal-load), and M_C has been set to unity. x_c is approximately set to 500 [mm], the half width of the component.

Figures 6.2 and 6.3 describe the log scaled hazard rate curve for $q = 1.0 \times 10^{-4}$ and 0.5×10^{-4}, respectively. The solid and interrupted lines represent $qh^{(O)}(n)$ and $(1-q)h^{(N)}(n)$, respectively. The former means that the failure is caused by the overload at the n-th cycle, and the latter by the normal load. Then, the sum of both constitutes the hazard rate that overloads are superimposed. For the comparison, the hazard rate in case of no overloads is also put as the dotted line on the both figures.

These figures suggest that in the early stage of low hazard rate the failure caused by overloads is rather important than the retardation effect of crack growth. Relating to the assumption of the Poisson arrival of overloads, $qh^{(O)}(n)$ changes slowly. On the other hand, $(1-q)h^{(N)}(n)$ changes rapidly, corresponding to the crack growth. In this example, the occurrence rate of overloads, q, has a significant effect on the retardation, while the effect of q on the overload failure seems to be canceled out by the retardation.

The reliability function curves are described as a log scaled $1-R(n)$ in the Fig.4, in which solid, interrupted, and dotted lines correspond to $q = 0.5 \times 10^{-4}$, $q = 1.0 \times 10^{-4}$, and no overloads, respectively.

Random variable	Type of distribution	Mean value	Coefficient of variation
X_1	special type of Eq.(27)	—	—
X_2	shifted exponential	2.5	0.8
X_3 (normal load)	shifted Rayleigh	70	0.5
X_3 (overload)	Weibull	140	0.2
X_4	Weibull	515	0.05
X_5	Weibull	587	0.05
X_6	Weibull	1641	0.1

Table 6.1. Types of distribution and values of parameters.

ε	1.0×10^{-11}
λ	0.5
n_z	1000
α_0	1.0
ρ	1.7
ξ	0.167
σ_c	0.2

Table 6.2. Values of parameters of special distribution.

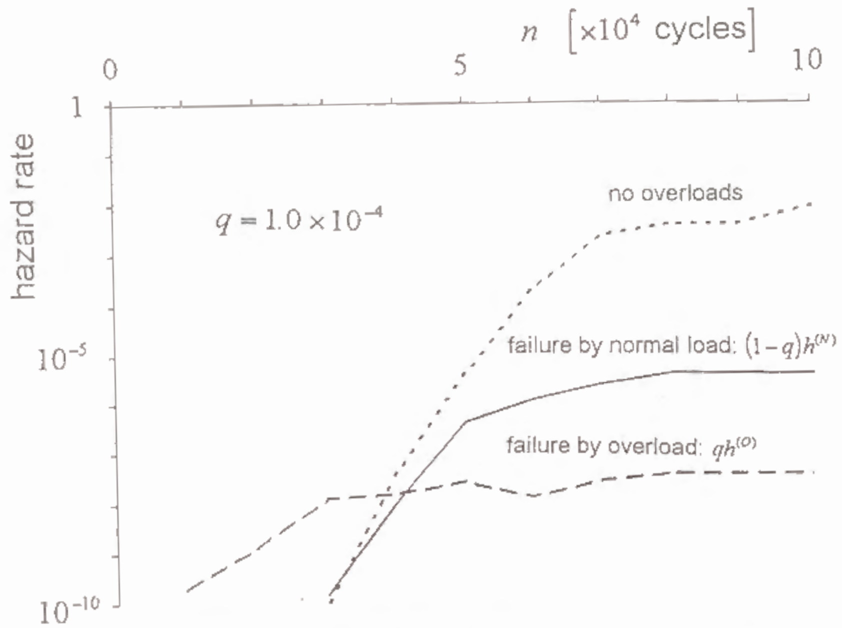

Figure 6.2: The hazard rate for $q = 1.0 \times 10^{-4}$.

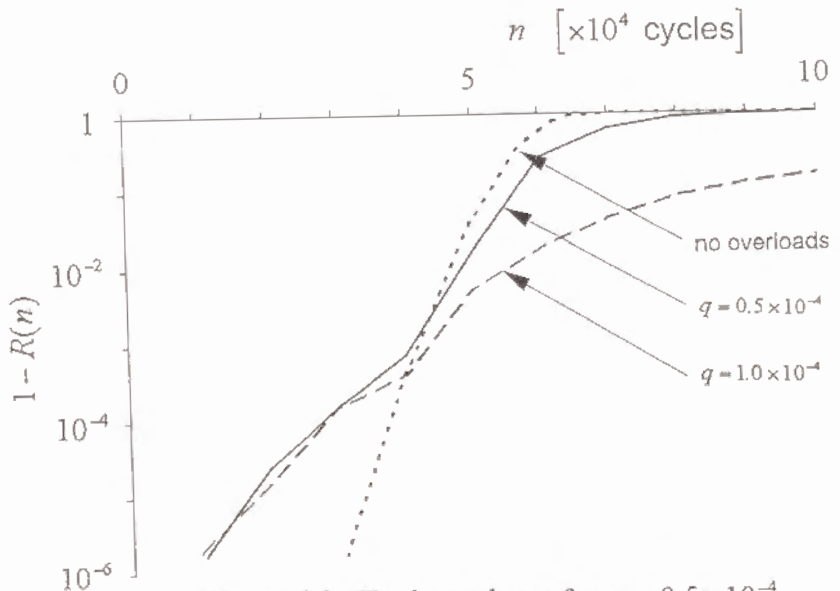

Figure 6.3: The hazard rate for $q = 0.5 \times 10^{-4}$.

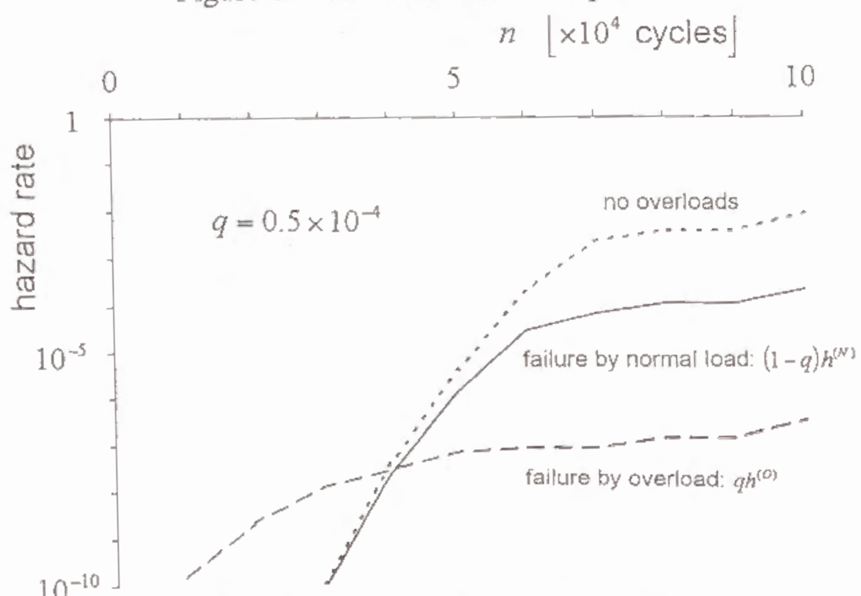

Figure 6.4: The reliability function.

Figure 6.4 also shows the superiority of the overload failure in the high reliability region and the diffusive effect of the life distribution by the retardation of crack propagation. These results suggest us that the extension of the fatigue life should not be expected in the high reliability region and that the retardation effect brings the more uncertainty on the life estimation.

6.6. Conclusions

It is shown that the hazard rate and reliability function for the fatigue failure is discussed on the assumption that the random overloads are superimposed at random timing. The retardation effect for crack propagation is expressed as a distribution of total delay time according to Arone's model. Arone's method does not treat the retardation effect with the crack propagation mechanism, but it is sufficient to the practical purpose if it is combined with the time variant distribution of the crack length. In order to estimate the risk of overloads, the Burdekin-Stone's two criteria approach is efficient for numerical failure analysis. Though the resulting expressions are not always easy to calculate, the cost of CPU time for the computation is kept in comparatively small level with the aid of the importance sampling simulation. The actual CPU time was 2500 seconds in average per one point in use of FACOM M-770/8. For the practical purpose, it is sufficient to evaluate the failure probability for overloads and to neglect the retardation effect in the high reliability region, because the delay of the failure become effective near the last of its life.

References

[1] Arone, R., Fatigue crack growth under random overloads superimposed on constant-amplitude cyclic loading, *Engng Fracture Mech.*, **24** (1986), 223-232.

[2] Arone, R., On retardation effects during fatigue crack growth under random overloads, *Engng Fracture Mech.*, **27** (1987), 83-89.

[3] Tsurui, A. and H. Ishikawa, Application of the Fokker-Planck equation to a stochastic fatigue crack growth model, *Structural Safety.*, **4** (1986), 15-29.

[4] Tanaka, H. and A. Tsurui, Reliability degradation of structural components in the process of fatigue crack propagation under stationary random loading, *Engng Fracture Mech.*, **27** (1987), 501-516.

[5] Larsson, H. and J. Bernard, Fracture of longitudinally cracked ductile tubes, *Int. J. Press. Vess. Piping*, **6** (1978), 223-243.

[6] Tsurui, A., J. Nienstedt, G. I. Schuëller and H. Tanaka, Time variant structural reliability analysis using diffusive crack growth models, *Engng Fracture Mech.*, **34** (1989), 153-167.

[7] Bourgund, U. and C. G. Bucher, Importance sampling procedure using design points, A User's Manual, Report No. 8-86 (1986), Institut für Mechanik, Univ. Innsbruck.

[8] C.E. Feddersen, Evaluation and prediction of the residual strength of center cracked tension panels, *ASTM STP* **486**, 50-78 (1970).

Chapter VII

Summary and Conclusions

In this study, some topics on the structural reliability analysis were investigated by applying the Markov approximation method, which is developed by Tsurui and Ishikawa, to the random growth of the fatigue crack. The topics are concerned with the problems which are in-service inspection of the structure, multiple site damage, LBB for the piping systems, and the retardation of the crack growth.

In chapters 2 and 3, we saw how the reliability of the structural component is assured by the in-service inspections with two typical policies. One is the repeated inspection with fixed intervals as is seen in chapter 2, and the other is history-dependent inspection given in chapter 3. The reliability estimation method for each inspection policy is established in the stochastic fracture mechanics, and the results pointed out that the randomness in the crack growth has a considerable effect, and that the inspection intervals have significant effects on the reliability. One of interesting results is that the optimal inspection does not always mean that the intervals become shorter as those of earlier inspection. The optimal inspection plan is strongly dependent on the probability for the detection of the cracks as well as the initial distribution of the crack length. On the other hand, in the history-dependent inspection, the first inspection interval is important on raising the reliability. The result suggests that one should pay attention to the quality of the inspection rather than the times of the inspection.

In chapter 4, the fracture from the multiple site damage is investigated in consideration of the interaction of the cracks. The distribution function for the two collinear cracks are evaluated by the combination of the analytical approximation and the numerical calculation. It is observed that the failure probability is sensitive to the initial distribution of the crack length. This result is expected to become the starting point of the reliability analysis for MSD problem.

In chapter 5, the LBB problem is discussed from the view point of the probabilistic fracture mechanics. The probability of occurrence of LBB is expressed in the closed form, which enables to evaluate the LBB probability up to very small value to be required in the reliability assessment on reactor technologies.

Finally, in chapter 6, the random crack growth in which the retardation effect due to the random overload is taken in to account is investigated on the basis of two-criteria approach. The mechanism of the retardation is not specified in this study, but the distribution of the crack length is evaluated by means of the distribution function for the retardation time. The method to evaluate the hazard rate is proposed in this chapter, and the reliability function also is presented in the integral form in a multi-dimensional space. The example is calculated through the importance sampling simulation in order to observe which of the two failure mode contributes to the hazard rate or the failure probability.

Throughout this work, the most emphasized point is a randomness in the crack growth process. In the many previous studies on the structural reliability by different authors, these studies were not treated as a stochastic model for reason of the complexity of explosive stochastic process in fatigue crack growth. However, the Markov approximation method on the crack growth process enables the analytical derivation of the distribution function of the crack length, which can be applied in the several problems on structural reliability analysis as seen in the previous chapters. The remaining subject is a numerical simulation for the crack growth model as a stochastic process, in order to verify the validity of the density function of the crack length which is approximately given by the Fokker-Planck equation. Especially, it will be necessary to develop an efficient method to simulate the multi dimensional fatigue crack process from the view point of the first passage problem in a stochastic field. For this purpose, one of the effective methods is to extend the importance sampling procedure to dynamic process.

In future, as we get more knowledge on the statistical properties in the crack growth, the approach from stochastic theory will become more important in the structural reliability analysis, because of the increasing requirement of structural safety.

Publication list

[1] A. Tsurui, H. Tanaka and T. Tanaka, Probabilistic analysis of fatigue crack propagation in finite size specimens, *Probabilistic Engineering Mechanics*, **4** (1987) 120-127.

[2] A. Tsurui, A. Sako and T. Tanaka, Effect of repeated inspections on Structural reliability degradation, *Journal of the Society of Materials Science, Japan* (in Japanese), **39** (1990) 748-752.

[3]* T. Tanaka and A. Tsurui, Reliability assurance for fatigue crack growth by repeated in-service inspections, *Structural Safety*, **9** (1991) 305-314.

[4] A. Tsurui, T. Tanaka and H. Tanaka, A stochastic fracture mechanical approach to LBB assessment for pipings, *Proceedings of SMiRT 11*, Vol.M (1991) 259-264.

[5] H. Tanaka, A. Tsurui and T. Tanaka, Hazard rate of structural components under fatigue environment including random overloads, *Proceedings of the 6th International Conference of Mechanical Behavior of Materials*, Vol.1 (1991), 639-644.

[6]* A. Tsurui and T. Tanaka, On history-dependent inspection policy for stochastic fatigue crack propagation, *Journal of the Society of Materials Science, Japan* (in Japanese), **41** (1992) 1025-1029.

[7]* T. Tanaka, A. Yamane A. Tsurui and H. Tanaka, A probabilistic approach to the random propagation of two collinear cracks, *Journal of the Society of Materials Science, Japan* (in Japanese), **42** (1993) 1400-1405.

[8]* A. Tsurui, H. Tanaka and T. Tanaka, An analytical method for leak before break assessment based upon stochastic fracture mechanics, *Nuclear Engineering and Design*, **147** (1994) 171-181.

[9]* T. Tanaka, Reliability analysis of structural components under fatigue environment including random overloads, *Engineering Fracture Mechanics*, to appear.

[10] C. Ihara, T. Yanagi and T. Tanaka, Estimation of creep life for metals based on damage mechanics, *Engineering Fracture Mechanics* **39** (1991), 887-893.

* The papers on which the present thesis based.